Math 630 Comprehensive Examination 8/23/2017

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Calculate $\|I\|_F$ and $\|I\|_2$, where I is the $n \times n$ identity matrix, and notice that they are different, (b) Use the Cauchy-Schwarz inequality to show that for all $A \in \mathbb{R}^{n\times n}$, $\|A\|_2 \leq \|A\|_F$.

2. Consider the system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$.

   (a) Describe the three possible cases for existence and uniqueness of a solution of the linear system. Give criteria on A, b that distinguish each case.

   (b) Let x_{LS} be a minimizer of the least squares functional, that is let

   $$\|Ax_{LS} - b\|_2 = \min_x \|Ax - b\|_2 .$$

   (i.) Does x_{LS} always exist? Explain your answer.

   (ii.) Give conditions on A, b such that x_{LS} is unique.

   (iii.) In the case of a unique solution, give an expression for the least squares solution x_{LS}.

   (iv.) If there is an infinite number of solutions to the least squares problem, find the solution of minimal norm.

   (c) The minimal norm solution can be computed by using the singular value decomposition (SVD) of A. Define the singular value decomposition and show how it can be used to compute the minimal norm least squares solution.

3. We say that two matrices $A, B \in \mathbb{C}^{n\times n}$ are *unitarily similar* if there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $B = U^{-1}AU$.

   (a) Show that if U is unitary, then $\|U\|_2 = 1$ and $\kappa_2(U) = 1$.

   (b) Show that if A and B are unitarily similar, then $\|A\|_2 = \|B\|_2$ and $\kappa_2(A) = \kappa_2(B)$.

   (c) Suppose $B = U^*AU$, where U is unitary. Show that if A is perturbed slightly, then the resulting perturbation in B is of the same magnitude. Specifically, show that if $B + \delta B = U^*(A + \delta A)U$, then $\|\delta B\|_2 = \|\delta A\|_2$.

4. (a) Show that if $x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$ be obtained by an exact line search, then

   $$\alpha_k = \frac{p^{(k)T} r^{(k)}}{p^{(k)T} A p^{(k)}},$$

   where $r^{(k)} = b - Ax^{(k)}$. (Hint: consider $g(\alpha) = J(x^{(k)} + \alpha p^{(k)})$.)

(b) Recall that in descend methods like conjugate gradients the residuals are related by the recursion $r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)}$. Let $x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)}$ be obtained from an exact line search. Show that then $r^{(k+1)} \perp p^{(k)}$ and $e^{(k+1)} \perp_A p^{(k)}$.