Math 630 Comprehensive Examination 8/24/2016

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Consider the system of linear equations $A\,x = b$ given by

$$A = \begin{bmatrix} 1 & 2 \\ 2 & \alpha \end{bmatrix}, \quad b = \begin{bmatrix} \beta \\ 1 \end{bmatrix}$$

with real constants α and β.

   (a) For which values of α and β does the system $A\,x = b$ have (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions? Simplify your terms. Summarize your results clearly and justify your answers.

   (b) In light of your analysis at (a), specify for each of the cases (i)–(iii) which of the following factorizations can be used to solve the system in a meaningful way (include least-squares solutions in the discussion): LU, Cholesky, QR.

   (c) In the case (i) of a unique solution, calculate the solution x as function of α and β. In the case (ii) of no solution find the least-norm solution of the associated least-squares problem using any method you prefer.

2. Assume A is nonsingular, $\|\delta A\| \,/\, \|A\| < 1/\kappa(A)$, $b \neq 0$, $Ax = b$, and

$$(A + \delta A)\,(x + \delta x) = b + \delta b.$$

   (a) Show that

$$\delta x = A^{-1}\left(\delta b - \delta A(x + \delta x)\right)$$

   and

$$\frac{1}{\|A\|} \leq \frac{\|x\|}{\|b\|}.$$

   (b) Use part (a) to show that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - \kappa(A)\,\frac{\|\delta A\|}{\|A\|}}.$$

3. Consider the simple iteration

$$x_{n+1} = x_n + M^{-1}(b - Ax_n) \tag{1}$$

for solving the linear equation $Ax = b$, with

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}.$$

(a) If x^* is the solution of the system and $e_n = x^* - x_n$ is the error at the nth step, show that

$$e_n = (I - M^{-1}A)^n e_0 ,$$

and give a necessary and sufficient condition for x_n to converge to x^* for **every** initial guess x_0.

(b) For $x_0 = [0, 0]^T$ compute the iterates x_1, x_2. Using your argument at (a) show that the iteration (1) converges to the solution x^*.

(c) Decide whether the Gauss-Seidel iteration applied *ad litteram* to the linear system $Ax = b$ converges or not. Can you relate iteration (1), with the matrices defined as above, with the Gauss-Seidel method?

4. (a) Show that if $P \in \mathcal{M}_n$ is an orthogonal (symmetric) projection ($P^2 = P$ and $P = P^T$) then $Q = I - 2P$ is an orthogonal matrix. Also show that the pseudo-inverse of P is P itself.

(b) Show that if $A \in \mathcal{M}_{m,n}$ has rank n then

$$\|A(A^TA)^{-1}A^T\|_2 = 1 .$$

Hint: Begin with the SVD of A.