Math 620 Comprehensive Examination 8/24/2016

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Apply Newton's method to the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0, \end{cases}$$

with the root $\alpha = 0$. What is the behavior of the iterates? Do they converge, and if so, at what rate?
(b) Do the same as in (a), but with

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & x \geq 0, \\ -\sqrt[3]{x^2}, & x < 0. \end{cases}$$

2. Consider the fixed point iteration method $x_{n+1} = g(x_n)$.

   (a) State the necessary conditions for existence and uniqueness of a fixed-point $x = \alpha$, and deduce the criteria that determines the order of convergence.

   (b) Consider instead the fixed-point iteration

$$x_{n+1} = G(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n}.$$

   Show that if α is a fixed-point of $g(x)$, then it is also a fixed point of $G(x)$.

   (c) Consider the function $g(x) = x^2$, and deduce the convergence properties for both fixed-point methods around the roots $x = 0$ and $x = 1$.

3. Consider the function $f(x) = x^4 - 2x^3 + 1$.

   (a) Calculate the interpolating polynomial $p_2(x)$ for data using the nodes $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Simplify the polynomial to standard form. Use the error theorem for polynomial interpolation to bound the error $|f(x) - p_2(x)|$ on the interval $[-1, 2]$. Is this bound realistic?

(b) Determine the conditions satisfied by the coefficients of the linear polynomial $\tilde{p}_1(x) = a_0 + a_1 x$ that solves the least-squares minimization problem

$$\min_{p \in \mathcal{P}_1} \int_{-1}^{2} |f(x) - p(x)|^2 \, dx \, .$$

(No need to solve the resulting system).

(c) Without explicitly computing the polynomial $\tilde{p}_1$ at (b), show that there are two distinct points $\xi_0, \xi_1 \in (-1, 2)$ so that $\tilde{p}_1$ is the interpolation polynomial of f at ξ_0, ξ_1.
Hint: Begin by showing that if $f - \tilde{p}_1$ has no root, then one of the orthogonality conditions defining $\tilde{p}_1$ is not satisfied. Proceed similarly to show that $f - \tilde{p}_1$ has a second root.

4. (a) Identify the coefficients A, B, C in the formula

$$D[y](x) = Ay(x - 2h) + By(x - h) + Cy(x) \quad (h \neq 0) \tag{1}$$

so that for any sufficiently smooth function y

$$y'(x) = D[y](x) + O(h^p)$$

with the highest possible approximation order p, which needs to be determined.

(b) Use formula (1) to derive a two-step method of the form

$$y_{n+1} = \sum_{j=0}^{2} a_j y_{n-j} + h \sum_{j=-1}^{2} b_j f(x_{n-j}, y_{n-j}) \tag{2}$$

for solving the initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0$$

on a grid $x_0, x_1, x_2, \ldots$ with uniform spacing $x_{n+1} - x_n = h$.
Hint: replace the exact derivative $y'(x_{n+1})$ in the differential equation with the expression from the numerical differentiation formula (1).

(c) Discuss the consistency, stability, and convergence properties of your method at (b).