Math 620 Comprehensive Examination 1/21/2016

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded. Note that each sub-problem is worth the same number of points.

1. (a) Let $f(x)$ be a function satisfying $|f^{(i)}(x)| \leq i!2^i, i = 1, \ldots n+1$, and let $p(x)$ be its polynomial interpolant throught the points $x_0, x_1, \ldots x_n$, all of which lie in $[0, \frac{1}{2}]$. Show that for any $x \in [0, \frac{1}{2}]$, we have

$$|f(x) - p(x)| \leq 1.$$

(b) Let $x_1, x_2, \ldots x_n$ be distinct. Let $l_1(x), l_2(x), \ldots l_n(x)$ be the Lagrange polynomials corresponding to $\{x_i\}$. Show that

$$l_1(x) + l_2(x) + \cdots + l_n(x) = 1.$$

(c) The temperature T is calculated every Friday at 10 AM for a full year at BWI airport to give a set of values (i, T_i), where $i = 1, 2, \ldots 52$ denotes the week. The polynomial interpolant $p(x)$ satisfying $p(i) = T_i, i = 2, 4, \ldots 52$ (even only) is constructed. Suppose now that $p(x)$ is used to estimate the temperature when i is odd (e.g. you calculate $p(5)$ or $p(7)$). Do you expect this to be a good approximation for the actual temperature T_i or not? Discuss.

2. (a) Identify the numbers c, w_1, w_2 so that the formula

$$Q(f) = w_1 f(c) + w_2 f(1)$$

approximates the integral

$$\int_0^1 x f(x) dx$$

with the highest possible degree of precision.

(b) For the formula derived at (a) find an upper bound for the error

$$\left| \int_0^1 x f(x) dx - Q(f) \right| .$$

(Hint: approximate f with a polynomial of appropriate degree, then integrate to estimate the error.)

3. The *trisection* method is a modification of the bisection method used to find a root of $f(x) = 0$ over a bracketing interval $[a, b]$, where, instead of dividing the interval into two at each step, we divide it into three:

(T1) Define $c = (2a + b)/3$, $d = (a + 2b)/3$.

(T2) If $\text{sign}(f(b)) \cdot \text{sign}(f(d)) \leq 0$, set $a = d$
else if $\text{sign}(f(d)) \cdot \text{sign}(f(c)) \leq 0$, set $a = c, b = d$
else set $b = c$.

(In other words, we find the first subinterval (from the right) which must contain a root, and take it to be our new bracket)

(T3) If $b - a \leq 2\varepsilon$, accept $\xi = (a + b)/2$ as the approximation to the root and stop. Otherwise, go back to step (T1)

(a) Count the above cycle as one step. How many such steps (call this number n) will be required to ensure that the error is less than or equal to ε?

(b) On an average, do you expect the trisection method to be more computationally efficient than the bisection method or not? Justify. You should assume the root has equal probability of being in any of the possible sub-intervals at each step. (Also, assume that all the computational work is concentrated in functional evaluations, and other steps are negligible.)

4. (a) Explain how floating-point numbers with a binary base are represented in computers, that is, how is a block of 32 bits or 64 bits in computer memory is related to the real number being represented. For your description, pick the example of one number system concretely, such as single-precision IEEE standard, double-precision IEEE standard, or similar; state which system you are describing.

(b) Assume a mathematical representation for normalized computer numbers of the abstract form $\tilde{x} = (-1)^s(1.b_1b_2 \ldots b_{p-1})_2 2^E$ with $s \in \{0, 1\}$, $b_j \in \{0, 1\}$, $j = 1, \ldots, p-1$, and $E_{\text{min}} \leq E \leq E_{\text{max}}$. Derive and explain the formulas for the (i) smallest normalized number, (ii) largest normalized number, and (iii) machine epsilon.

(c) Discuss how there is a trade-off between the range of a number system (the size of the largest number; larger is better) and the relative accuracy of the number system (the size of machine epsilon; smaller is better).