Math 620 Comprehensive Examination 8/20/2014

Name:

**Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.**

1. (a) Consider the iteration

$$x_{n+1} = g(x_n) \quad n \geq 0.$$

State conditions under which this iteration will converge to $\alpha$, a fixed point of $g$, with order $p$.

(b) Use the above to show that (under appropriate conditions) $p$ is exactly 2 when Newton's method is used to find a root of the equation $f(x) = 0$.

(c) Consider the equation

$$x = 1 + hf(x)$$

with $f(x)$ continuous for all $x$. For $h = 0$, a root is $\alpha = 1$. Show that for all sufficiently small $h$, this equation has a root $\alpha(h)$.
HINT: Consider the image of the interval $[0, 2]$ under a suitable mapping.

2. (a) Let $\mathcal{P}_2$ =span$\{1, x, x^2\}$. Let $p \in \mathcal{P}_2$ be the minimax approximation to $f(x) = \cos x + \sin x$ on the interval $[0, 1]$. Show that

$$|f(x) - p(x)| \leq \frac{1}{3!} + \frac{1}{4!}$$

for all $x \in [0, 1]$.
HINT: You don't need to actually calculate any minimax approximation in this problem.

(b) Let $\mathcal{Q}$ =span$\{\sin x, \cos x, \cos 2x\}$. Let $q \in \mathcal{Q}$ be the minimax approximation to $f(x) = 2x - x^2$ on the interval $[0, 1]$. Show that

$$|f(x) - q(x)| \leq \frac{2}{3!} + \left(\frac{2}{3}\right)\left(\frac{17}{4!}\right)$$

for all $x \in [0, 1]$.
HINT: Again, you don't need to actually calculate any minimax approximation in this problem. The functions in $\mathcal{Q}$ can be thought of as their Taylor polynomials + remainder terms. What linear combination of functions in $\mathcal{Q}$ will therefore be a good approximation to $f(x)$?

3. (a) Find quadratic polynomials $H_0(x), H_1(x), H_2(x)$ such that for any function $f(x) \in C^{(2)}([0, 2])$, we have

$$H(x) = f(0)H_0(x) + f'(1)H_1(x) + f''(2)H_2(x)$$

satisfies the interpolating conditions

$$H(0) = f(0), H'(1) = f'(1), H''(2) = f''(2).$$

(b) **Using the above polynomials** $H_0, H_1, H_2$ (and not any other way), find $A, B, C$ such that the quadrature rule

$$\int_0^2 f(x)dx \approx Q(f) = Af(0) + Bf'(1) + Cf''(2)$$

is exact for all quadratics.

(c) Find the degree of precision of the above quadrature rule.

4. Consider the initial value problem

$$y' = 1 + x + y, \quad y(0) = 1.$$

Compute two steps of the Runge-Kutta method defined by the tableau

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

with constant step size $h = 0.1$.