Math 620 Comprehensive Examination 1/23/2014

Name:

**Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.**

1. (a) State the Contraction Mapping Theorem (CMT) for a bounded region in $\mathbb{R}^d$.

   (b) Consider the system

$$x = \frac{0.5}{1+(x+y)^2} \qquad y = \frac{0.5}{1+(x-y)^2} \ . \tag{1}$$

   Identify a region $D \in \mathbb{R}^2$ where the CMT applies and verify that the hypotheses hold.
   (Hint: it helps to first sketch the graph of the function $t \mapsto t/(1+t^2)$.)

   (c) Starting with the initial guess $(x_0, y_0) = (0,0)$ compute the iterates $(x_n, y_n)$, $n = 1, 2$ resulted from applying the fixed point methods to (1), and estimate the number of iterations needed for the $n^{\text{th}}$ iterate to satisfy $\|(x_n, y_n) - (x^*, y^*)\|_\infty \leq 10^{-8}$, where $(x^*, y^*)$ is the unique solution of (1) in your set $D$.

2. It is given that for each set of real numbers $x_0 < x_1 \cdots < x_n$ and function $f \in C^{n+1}(\mathbb{R})$ the unique Lagrange interpolation polynomial $p_n(x)$ interpolating $f$ at $x_0, \ldots, x_n$ satisfies

$$f(x) - p_n(x) = f[x, x_0, x_1, \ldots, x_n] \prod_{j=0}^{n} (x - x_j) \ ,$$

   where $f[x, x_0, \ldots, x_n]$ denotes the $(n+1)^{\text{th}}$ Newton divided difference of $f$. Assume $f[x,x] = f'(x), f[x,x,y] = (f(x,y) - f[x,x])/(y-x)$ for $x \neq y$.

   (a) Using the above result and assuming the continuity of the divided differences show that for $f \in C^3(\mathbb{R})$ and $x_0 < x_1$ there exists a unique quadratic polynomial $q$ so that

$$q(x_0) = f(x_0), \ q'(x_0) = f'(x_0), \ q(x_1) = f(x_1) \ ,$$

   and that for $x_0 \leq x \leq x_1$

$$|f(x) - q(x)| \leq \frac{M}{2}(x - x_0)^2 (x - x_1) \ , \tag{2}$$

   with $M = \max_{x_0 \leq x \leq x_1} |f^{[3]}(x)|$.
   (Hint: consider the Langrange polynomial of $f$ at $x_0, x_0 + \varepsilon, x_1$, write the error

formula and let $\varepsilon \to 0$; make sure to justify all the steps and to verify the needed properties.)

(b) Find the quadratic polynomial that satisfies

$$q(1) = 1, \ q'(1) = 1/2, \ q(4) = 2 \ ,$$

and use (2) to estimate $|\sqrt{1.5} - q(1.5)|$.

3. Suppose $\{\psi_0, \psi_1, \psi_2\}$ are orthogonal polynomials on $[0, 1]$ with weight $w(x)$ such that for $i = 0, 1, 2$, we have

$$\int_0^1 w(x)(\psi_i(x))^2 dx = i + 1.$$

Also, suppose the function $f \in C[0, 1]$ is such that

$$\int_0^1 w(x) f(x) \psi_i(x) dx = 2i^2 + 1.$$

(a) Find $r_2^*$, the best least squares quadratic approximation to $f$.

(b) What is $\|r_2^*\|_2 = \left(\int_0^1 w(x)(r_2^*(x))^2 dx\right)^{\frac{1}{2}}$?

(c) Find $\|f - r_2^*\|_2$ if $\|f\|_2 = 7$.

4. Consider the following numerical method for the initial value problem $y'(x) = f(x, y(x))$, $y(0) = Y_0$:

$$y_{n+1} = y_n + h[af(x_n, y_n) + bf(x_{n+1}, y_{n+1})] \ (n \geq 0), \quad y_0 = Y_0,$$

where we take $a = \frac{1}{3}$ and $b = \frac{2}{3}$. Suppose this method is used to find an approximate solution for the problem where $f(x, y) = \lambda y, Y_0 = 1$, and $\lambda < 0$.

(a) Show that for a fixed point $\bar{x} = nh$, if $n \to \infty$ (i.e. $h \to 0$) then $y_n \to y(\bar{x})$. (Hint: Derive an explicit formula for $y_n$.)

(b) Show that for $h$ fixed, if $n \to \infty$ then $y_n \to 0$.

(c) What would be the best choice of weights $a, b$ and why? (Hint: Think of how the method may be derived in terms of numerical quadrature.)