Math 630 Comprehensive Exam 8/22/2012

Name:

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded. You may use scientific calculators; however, programmable calculators, laptops, etc. are not allowed.

[1] Let $A \in M_n$ be a diagonalizable matrix, that is, there exists a nonsingular matrix $V \in M_n$ for which $V^{-1}AV = D$ with D being diagonal. Define the function

$$\|\cdot\|_V : \mathbb{R}^n \to [0, \infty), \quad \|x\|_V = \|V^{-1}x\|_2 \,.$$

(a) Show that $\|\cdot\|_V$ is a vector norm.

(b) Prove that the matrix norm $|||\cdot|||$ induced by the vector norm $\|\cdot\|_V$ has the property that $|||A||| = \rho(A)$, where $\rho(A)$ is the spectral radius of A.

[2]
(a) Explain **briefly** what the unit LU factorization of a matrix is and state a result regarding the existence of such a factorization.

(b) Consider the $n \times n$ matrix $A_n = \mathrm{diag}(-1, 2, -1)$, that is,

$$A_n = \begin{bmatrix} 2 & -1 & 0 \dots & 0 & 0 \\ -1 & 2 & -1 \dots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \dots & 2 & -1 \\ 0 & 0 & 0 \dots & -1 & 2 \end{bmatrix}.$$

Use your result at (a) to argue that A_n has a unit LU factorization (you may restrict your attention to $n = 1, 2, 3$, or use without proof some known properties of A_n). Find a unit LU factorization of A_3.

(c) Show that if a nonsingular matrix A has an LU factorization with L unit lower triangular, then the factorization is unique. You may use without proof that the product and the inverses of (unit) upper, respectively lower, triangular matrices are (unit) upper, respectively lower triangular.

[3] Let $c, d \in \mathbb{R}$ be so that $|c|^2 + |d|^2 \neq 0$, and consider the matrix

$$A = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} .$$

(a) Compute the pseudo-inverse A^+ of A. Hint: compute the singular values of A and the orthogonal matrices U, V for which $A = U\Sigma V^T$ by finding eigendecompositions of the symmetric matrices A^TA and AA^T.

(b) Let $b = [1, 1]^T \in \mathbb{R}^2$ and consider the least-squares problem

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|^2 . \tag{1}$$

Show that (1) has infinitely many solutions. Can (1) be solved using the normal equations?

(c) Find the least-norm solution of (1), that is, the minimizer $x_{\min}$ of

$$\min \{\|x\| \ : \ x \text{ solves (1)}\} . \tag{2}$$

[4] Consider the matrix $A = \begin{bmatrix} -1 & -4 \\ 0 & 3 \end{bmatrix}$.

(a) Starting with initial guess $q_0 = [1, 1]^T$, compute three steps of the power method to find eigenvector approximations q_1, q_2, q_3.

(b) Use the Rayleigh quotient to compute the approximations λ_1, λ_2, λ_3 for each of the eigenvector approximations q_1, q_2, q_3, respectively.

(c) Determine the true value of λ that is being approximated by the power method. Compute the errors of the three approximations λ_1, λ_2, λ_3 with the true value λ. Do you observe convergence of the method?