Math 612

PHD COMPREHENSIVE EXAM
(Ordinary Differential Equations)
January, 2017

Do any three of the four problems below. Show all steps and justify your answers. Each problem is worth ten points.

1. Two trajectories $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of the four dimensional linear system $\dot{\mathbf{x}} = A_{4\times 4}\,\mathbf{x}$ are given by
$$\mathbf{x}_1(t) = te^{-2t}\mathbf{v}_1 + e^{-2t}\mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{x}_2(t) = e^{-t}\mathbf{v}_4, \quad (t \in \mathbb{R}),$$
where $\mathbf{v}_1, \cdots, \mathbf{v}_4$ are constant, nonzero vectors in $\mathbb{R}^4$.
   (a) Prove that $\mathbf{v}_i, i = 1, \cdots, 4$ are generalized eigenvectors of A and identify the corresponding eigenvalues.
   (b) Prove that $\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$ is a basis of $\mathbb{R}^4$.
   (c) What can you conclude about the limit as $t \to \infty$ of general trajectories of the system $\dot{\mathbf{x}} = A_{4\times 4}\,\mathbf{x}$? Discuss all possible initial conditions.

2. Consider the initial value problem on $E = \mathbb{R}$ given by
$$\dot{x} = -Cx + \frac{|x|}{1+|x|}, x(0) = x_0 \in \mathbb{R},$$
and where $C > 0$. Explain why the ODE has a locally unique solution. Show that the maximal interval of existence of the solution is $\mathbb{R}$. Justify your answer and state clearly what facts, if any, you are using.

3. Let $f : E \to \mathbb{R}^n$ be a locally Lipschitz vector field, where $E \subset \mathbb{R}^n$ is open. Let $\bar{x}$ be an equilibrium of f.
   (a) Provide the definitions of **Lyapunov stability** and (local) **asymptotic stability** of the equilibrium $\bar{x}$.
   (b) Prove that if $\bar{x}$ is asymptotically stable for f then it is not Lyapunov stable for $-f$. (HINT: There exists $x_0 \in E$ such that $\varphi_t(x_0) \to \bar{x}$ as $t \to \infty$ where φ_t is the flow of f. Now relate the flow ψ_t of $-f$ to φ_t.)
   (c) Is it true that if $\bar{x}$ is Lyapunov stable for f then it is not Lyapunov stable for $-f$? Prove or provide a counter example.

4. Let $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz vector field.
   (a) Provide the definition of the **omega limit set** of a trajectory Γ_x of the vector field that passes through a point $x \in E$.
   (b) Let $\bar{x} \in E$ be a Lyapunov stable equilibrium point of f. Suppose that $\omega(\Gamma_x)$, the omega limit set of a trajectory Γ_x passing through x, contains $\bar{x}$. Prove that $\varphi_t(x) \to \bar{x}$ as $t \to \infty$ where φ_t is the flow of f.