MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
January 2014

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 (a) Provide the sequential criterion for compactness of a set in a metric space.

(b) Let (M_i, d_i) be metric spaces for $i = 1, 2$. Let $M = M_1 \times M_2$ and define the metric d on M by

$$d(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Define the projection $\pi : M \to M_1$ by $\pi(x) = x_1$ where $x = (x_1, x_2)$.

Explain why direct images of compact sets under π are compact.

(c) With π as defined above, prove that inverse images of compact sets under π are also compact provided M_2 is compact.

Q2 (a) State the definition of connectedness of a set in a metric space.

(b) Show that (in any metric space), the closure of a connected set is connected. Prove or disprove that the interior of a connected set is connected.

(c) In $\mathbb{R}^n$, let $\mathbb{R}^n_+$ denote the nonnegative orthant (consisting of vectors with all components nonnegative). Show that $\mathbb{R}^n_+$ and $\mathbb{R}^n_+ \setminus \{0\}$ are connected. (Hint: Use convexity.)

Q3 (a) State a necessary and sufficient condition for a set to be compact in the metric space $C([0, 1])$ consisting of all real valued continuous functions on $[0, 1]$ endowed with the supremum norm metric.

(b) Prove or disprove that the closed unit ball $K = \{f : \|f\| \leq 1\}$ is compact in $C([0, 1])$.

(c) Consider the mapping $T : C([0, 1]) \to C([0, 1])$ defined by

$$(Tf)(s) = \int_0^1 (s + t)f(t)\, dt.$$

Show that $T(K)$ is equicontinuous and bounded (in the sup-norm metric).

Q4 Consider the series

$$\sum_{n=1}^{\infty} \frac{1-e^{-nx}}{1+n^2x^2},$$

on $[0, \infty)$.

(a) Show that the series converges uniformly on $[\delta, \infty)$ for every $\delta > 0$.

(b) Show that the series converges pointwise on $[0, \infty)$.

(c) Show that the series does not converge uniformly on $[0, \infty)$.

Q5 (a) Provide the definition of the (Frechet) derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are normed vector spaces (possibly infinite dimensional).

(b) Let $C([0,1])$ be the space of continuous real valued functions on $[0,1]$ endowed with the supremum norm. Define $F : C([0,1]) \to \mathbb{R}$ by

$$F(f) = \frac{1}{2}\int_0^1 (f(x))^2 dx - \frac{1}{2}\int_0^1 f(\sqrt{x})dx,$$

for all $f \in C([0,1])$.
Show directly from the definition that the derivative of F at $f \in C([0,1])$ is given by

$$DF(f)(g) = \int_0^1 f(x)g(x)dx - \frac{1}{2}\int_0^1 g(\sqrt{x})dx, \quad \forall g \in C([0,1]).$$

(c) For the F defined above, show that $DF(f)$ is zero if and only if f is given by $f(x) = x$.