Masters Comprehensive Exam in Matrix Analysis (Math 603)
August 2014

Do any **three** problems. **Show all your work**. Each problem is worth 10 points.

1. R^n denotes the Euclidean n-space with the usual inner product. Suppose $u \neq 0$, v, and w are (column) vectors in R^n.

(a) Show that uu^T is a real symmetric positive semidefinite matrix of rank one.

(b) If, for each x in R^n,
$$\langle u, x\rangle = 0 \Rightarrow \langle v, x\rangle = 0,$$
show v is a scalar multiple of u.

(c) If $uu^T = vv^T + ww^T$, show that $\langle u, x\rangle^2 = \langle v, x\rangle^2 + \langle w, x\rangle^2$ and (hence) v and w are multiples of u.

2. Given column vectors u and v in R^n, consider the matrix uv^T whose (i, j)th entry is $u_i v_j$. Let $e_1, e_2, \ldots, e_n$ denote the standard unit vectors in R^n.

(a) If $a_1, a_2, \ldots, a_n$ are nonzero column vectors in R^n, show that the matrices $\{a_i e_i^T : i = 1, 2, \ldots, n\}$ are linearly independent in $R^{n\times n}$ (the space of all real $n \times n$ matrices).

(b) Show that $R^{n\times n} = span\{uv^T : u, v \in R^n\}$.

3.

(a) If A is a nonsingular matrix in $R^{n\times n}$ and if u and v are column vectors in R^n, then show that $\det(A + uv^T) = \det(A)(1 + v^T A^{-1} u)$.

(b) Let $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a diagonal real matrix such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, and let v be a column vector in R^n with each entry being nonzero. Prove that if $\alpha \neq 0$ in R, then each λ_i is not an eigenvalue of $D + \alpha vv^T$.

4. Let A and B be $n \times n$ matrices such that $A = A^2$, $B = B^2$, and $AB = BA = 0$.

(a) Prove that $rank(A + B) = rank(A) + rank(B)$.

(b) Prove that $rank(A) + rank(I_n - A) = n$.

5. Let λ_1 and λ_2 be two distinct eigenvalues of a matrix A whose eigenspaces are E_1 and E_2 respectively. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases of E_1 and E_2 respectively.

(a) Show that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$, and $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $E_1 + E_2$.

(b) Show that if A is a normal matrix, then $E_1 \perp E_2$.