PhD COMPREHENSIVE EXAM IN
ORDINARY DIFFERENTIAL EQUATIONS
August 2013

Do any 3 of the following 4 problems. Show all work. Each problem is worth ten points.

Q1. Consider a linear system $\dot{x} = Ax$ where A is a 6×6 real matrix. Suppose

$$J = \begin{bmatrix} J_s & 0 \\ 0 & J_c \end{bmatrix},$$

is the (complex) Jordan form of A where J_s is $n_s \times n_s$ and has all eigenvalues with negative real part and J_c is $n_c \times n_c$ and has all eigenvalues with zero real part. It is thus assumed A has no eigenvalues with strictly positive real parts. For each of the following scenarios describe the stability of origin (asymptotically stable, stable but not asymptotically stable, unstable).

(a) $n_s = 6, n_c = 0$.

(b) $n_s = 5, n_c = 1$.

(c) $n_s = 4$, $n_c = 2$ and A is non-singular.

(d) $n_s = 4$, $n_c = 2$ and the null space of A has dimension 1.

(e) $n_s = 3, n_c = 3$ and i is an eigenvalue.

Q2. Given the ODE

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 - x_1^2, \\ \dot{x}_2 &= -x_2 + x_1^2 + x_2^2, \end{aligned}$$

compute the one dimensional center manifold through the origin $(0, 0)$ and determine the stability (asymptotically stable, stable but not asymptotically stable, unstable) of the equilibrium at origin.

Q3. You are given the ODE in $\mathbb{R}^n$:

$$\dot{x}(t) = \frac{Ax(t)}{1 + |x(t)|^2},$$

where A is a real $n \times n$ matrix and $|.|$ denotes the 2-norm in $\mathbb{R}^n$.

(a) Show that solution exists for all $t \in \mathbb{R}$ for any given initial condition $x_0 \in \mathbb{R}^n$.
HINT: Compute the partials.

(b) Suppose eigenvalues of A all have negative real parts. Decide the stability (asymptotically stable, stable but not asymptotically stable, unstable) of the equilibrium at origin.

Q4. Consider the 2D equation:

$$\dot{x} = -x^2 + \frac{1}{1+y^2},$$
$$\dot{y} = -y + x,$$

with initial condition $(x(0), y(0)) = (x_0, y_0)$ where $x_0 > 0$ and $y_0 > 0$.

(a) Show that $x(t)$ shall not become negative.

(b) Let $[0, \beta)$ be the forward maximal interval of existence. First obbtain the bound $0 \le x(t) \le x_0 + t$ for all $t \in [0, \beta)$.

(c) Use above to obtain the bound

$$0 \le y_0 e^{-t} \le y(t) \le y_0 e^{-t} + x_0(1 - e^{-t}) + (t + e^{-t} - 1) \le y_0 + x_0 + t,$$

for all $t \in [0, \beta)$.

(d) Show that $\beta = \infty$.