Masters Comprehensive Exam in Matrix Analysis (Math 603)

January 2013

Do any **three** problems. **Show all your work**. Each problem is worth 10 points.

1. Let $\mathcal{A} = \{v_1, v_2, v_3\}$ be a basis in R^3.

(a) Show that $\mathcal{B} = \{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is also a basis of R^3.

(b) For the linear transformation $L : R^3 \to R^3$ defined by $L(v_1) = 2v_1$, $L(v_2) = 2v_2$ and $L(v_3) = 2v_3$, find the matrix representation of L with respect to the bases $\mathcal{A}$ and $\mathcal{B}$.

2. Let $T \in \mathcal{L}(V)$ be a linear operator on an n-dimensional real inner-product space $(V, \langle \cdot\,,\, \cdot \rangle)$ whose singular value decomposition is given by two orthonormal bases $(u_1, u_2, \ldots, u_n)$, $(v_1, v_2, \ldots, v_n)$ of V and singular values $\sigma_1 \geq \sigma_2 \geq \ldots, \sigma_n \geq 0$, such that

$$Tx = \sum_{j=1}^{n} \sigma_j \langle\, x\,,\, v_j\,\rangle u_j, \quad \forall\, x \in V.$$

(a) Prove that for any $m < n$ we have

$$\|Tx - \sum_{j=1}^{m} \sigma_j \langle\, x\,,\, v_j\,\rangle u_j\| \leq \sigma_{m+1}\|x\|, \quad \forall\, x \in V.$$

(b) What are the eigenvalues and eigenvectors of TT^*?

(c) Find an orthonormal basis for the null space of T, $Ker(T)$, and a basis of the range space of T, $Range(T)$ when $n = 10$ and $\sigma_7 > \sigma_8 = 0$.

3. Let A be an $m \times n$ matrix with rank m.

(a) Prove that there is an $n \times n$ orthogonal matrix Q and an $m \times m$ upper-triangular matrix R_1 with strictly positive diagonal entries such that

$$A^T = QR, \text{ where } R \text{ is the } n \times m \text{ matrix } R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}.$$

(b) Find an orthonormal basis for the column space of A^T, $Col(A^T)$ and an orthonormal basis for the nullspace of A, $Nul(A)$.

(c) Prove that for any $b \in I\!\!R^m$ the minimization problem

$$\min \|x\|$$
$$\text{such that: } Ax = b,$$

has a unique solution x^* and that $\|x^*\| = \|\left(R_1^{-1}\right)^T b\|$.

4.

(a) Let A be $m \times m$ and $\det A \neq 0$ prove that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B).$$

(b) If A, B, C and D are all $m \times m$ matrices and $AB = BA$, prove that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(DA - CB).$$

(Hint: Use part (a).)

5. Let $A = [a_{ij}]$ be a complex square matrix.

(a) If $tr(A)$ — trace of A — is the sum of all its eigenvalues, show that

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

(b) Show that if $A^n = 0$, then every eigenvalue of A is zero.

(c) Prove the converse in (b).