MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
January 2013

Do any three problems. Show all work. Each problem is worth ten points.

1. Let S be a nonempty closed subset of a metric space (M, d). For each $x \in M$, let
$$d(x, S) := \inf\ d(x, s),$$
where the infimum is taken over all s in S. Show the following:
   (a) $d(x, S) = 0$ if and only if $x \in S$.
   (b) As a function of x, $d(x, S)$ is a (Lipschitz) continuous function on M.
   (c) If K is another nonempty closed set in M that is disjoint from S, then the function
$$f(x) := \frac{d(x, S)}{d(x, S) + d(x, K)}$$
is a well-defined continuous function from M to the interval $[0, 1]$ with $f(x) \equiv 0$ on S and $f(x) \equiv 1$ on K.
   (d) If (M, d) is R^n with the usual metric (and S is a nonempty closed set), show that the infimum in the definition of $d(x, S)$ is always attained.

2. A function $f : R^n \to R^m$ is said to be proper if it is continuous and the following implication holds:
$$\text{For any sequence } x_k,\ \ ||x_k|| \to \infty \Rightarrow ||f(x_k)|| \to \infty.$$
   (a) Give an example of a proper function from R to R.
   (b) Show that a continuous function from R^n to R^m is proper if and only if inverse image of any compact set is compact.
   (c) If $f : R^n \to R$ is a proper function, show that $|f|$ attains its global minimum on R^n.

3. For a real variable x, consider the power series

$$\sum_{1}^{\infty} \frac{x^n}{n(n+1)}.$$

(a) Find the radius of convergence of the above power series.

(b) What is the interval of convergence of the given series?

(c) Justifying all steps, show that, in the interior of the interval of convergence,

$$(1-x)[2xy' + x^2y''] = x,$$

where y denotes the sum of the given power series with y' and y'' denoting the first and second derivatives of y respectively.

4. Let $\mathcal{F}$ be a family of real valued functions defined on a metric space (M, d).

(a) State the definition of *equicontinuity* for $\mathcal{F}$.

(b) Show that every member of an equicontinuous family is uniformly continuous. Show that the converse holds if $\mathcal{F}$ is a finite set.

(c) Let $g : [0, 1] \to R$ be continuous. For any natural number n and $x \in [0, 1]$, let

$$f_n(x) = g(x/n).$$

Show that $f_n(x) \to g(0)$ uniformly on $[0, 1]$. Is the family $\{f_n : n = 1, 2, \ldots\}$ equicontinuous?

5. (a) State the definition of (Fréchet) derivative of a function between two normed linear spaces.

(b) For the function $f : R^n \to R^n$ defined by

$$f(x) = (||x||^2 - 1)x,$$

show that the derivative is given by $Df(x) = (||x||^2 - 1)I + 2xx^T$, where I denotes the identity matrix and x^T denotes the transpose of the (column) vector x in R^n.

(c) When $n = 1$ solve the equation $Df(x) = 0$.

(d) When $n > 1$, show that $Df(x)$ is nonzero for every x in R^n.