Masters Comprehensive Exam in Matrix Analysis (Math 603)
August 2012

Do any **three** (out of five) problems. **Show all your work.** Each problem is worth 10 points.

(Q1) (a) Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ be a basis of an n dimensional vector space $\mathcal{S}$. Show that if a vector β in $\mathcal{S}$ has the property that it can be expressed as a linear combination of every $n-1$ vectors from $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$, then $\beta = 0$.

(b) Find the rank and nullity of the of the following matrix (as a linear transformation on R^3):

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

(Q2) Let A be a real $n \times n$ matrix.

(a) Show that $A = 0$ if $AA^T = 0$.

(b) Show that A is symmetric if and only if $A^2 = AA^T$.

(Q3) Let A be a $p \times n$ matrix and B be an $n \times m$ matrix. Denote $\mathcal{L}_0 = \{x \in R^m : \ ABx = 0\}$ and $\mathcal{L}_1 = \{y \in R^n : \text{ there exists } x \in \mathcal{L}_0 \text{ such that } y = Bx\}$.

(a) Show that $dim\mathcal{L}_1 = rank(B) - rank(AB)$.

(b) Show that for any $n \times n$ matrices $A, \ B$ and C, $rank(AB) + rank(BC) \leq rank(B) + rank(ABC)$.

(Q4) Let A be a complex $n \times n$ matrix satisfying the equation $I + A + A^2 = 0$.

(i) What are the eigenvalues of A?

(ii) Show that A and $I + A$ are invertible.

(iii) Express $\det(I + A)$ in terms of $\det(A)$.

(iv) Give an example of such a matrix.

(Q5) Let P be an $n \times n$ real matrix satisfying the conditions

$$P^T = P \quad \text{and} \quad P^2 = P.$$

(Such matrices are called projections.)

(i) Show that the eigenvalues of P are real. What are they?

(ii) Show that $x^T P x \geq 0$ for all $x \in R^n$.

(iii) If $Ker(P)$ and $Ran(P)$ denote, respectively, the kernal (=null space) and range of P, show that R^n is the direct sum of $Ker(P)$ and $Ran(P)$.