MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
August 2012

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 A real valued function f on a metric space (M, d) is said to be *upper semicontinuous* if for all $\lambda \in \mathbb{R}$, the set $\{x \in M \mid f(x) < \lambda\}$ is open in M.

(a) Provide the open cover definition of compactness of a set in M.

(b) Suppose M is compact and that $f : M \to \mathbb{R}$ is upper semicontinuous. Show that f is bounded above and attains a maximum value on M. (HINT: To show f is bounded above consider sets $\{x \in M \mid f(x) < n\}$ for $n \in \mathbb{N}$. To show attainment of maximum do a proof by contradiction by considering the supremum of the set of values attained by f.)

Q2 (a) Provide the definitions of *uniform convergence* of a sequence (f_n) of functions $f_n : A \to \mathbb{R}$ to a function $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$.

Rest of Q2 relates to the series of functions

$$\sum_{n=1}^{\infty} \frac{x^2 - nx}{n^3 + nx},$$

of a real variable x on the domain $[0, \infty)$.

(b) Show that the series converges pointwise on $[0, \infty)$. Does it converge uniformly on $[0, \infty)$? Justify your answer.

(c) Is the sum continuous on $[0, \infty)$? Justify your answer.

(d) Is the sum differentiable on $(0, \infty)$? Justify your answer.

Q3 For $n \in \mathbb{N}$ let $f_n : A \to \mathbb{R}$ where $A \subset \mathbb{R}$.

(a) State the definition of *equicontinuity* for the sequence (f_n) and state the *Arzela - Ascoli* theorem in this context.

(b) Let $A = [0, 1]$ and let f_n be defined by

$$f_n(x) = \frac{1}{(1 + \frac{x}{n})^n}.$$

Prove that f_n converges uniformly on A to some function f. What is f?

Q4 (a) State the *contraction mapping theorem* (also known as the Banach fixed point theorem).

(b) Suppose (M, d) is a compact metric space and for $n \in \mathbb{N}$ let $f_n : M \to M$ be contraction mappings. Suppose the sequence (f_n) converges *uniformly* on M to $f : M \to M$. Prove that f has at least one fixed point.

Q5 (a) Provide the definition of the derivative of a map $F : V \to W$ at $x \in V$ where V and W are (possibly infinite dimensional) normed vector spaces (over $\mathbb{R}$).

(b) Let $C([0, 1])$ be the Banach space of continuous functions from $[0, 1] \subset \mathbb{R}$ into $\mathbb{R}$. Let $F : C([0, 1]) \to \mathbb{R}$ be defined by

$$F(f) = \frac{1}{2} \int_0^1 (f(t))^2 dt - \left(\int_0^1 f(t) dt \right)^2,$$

for all $f \in C([0, 1])$. Prove *directly using the definition of derivative* that F is differentiable at every $f \in C([0, 1])$ and that the derivative is given by

$$DF(f)(g) = \int_0^1 f(t)g(t)dt - 2 \left(\int_0^1 f(t)dt \right) \left(\int_0^1 g(t)dt \right),$$

for all $f, g \in C([0, 1])$.

(c) Show that F has an extreme value at f if and only if f is the zero function. Is the extreme value a local minumum, a local maximum or neither? Explain.