Math 620 Comprehensive Examination

January, 2023

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) The three-point Gaussian quadrature rule on $[-1,1]$ (with weight 1) is given by

$$\int_{-1}^{+1} f(x)dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

Use a change of variable to find the nodes and weights for

$$\int_0^8 g(t)dt \approx w_1 g(t_1) + w_2 g(t_2) + w_3 g(t_3),$$

the corresponding rule on $[0,8]$.

(b) A certain quadrature rule for approximating $\int_a^b f(x)dx$ has error estimate given by

$$E(f) = C(b-a)^6 f^{(5)}(\eta)$$

where η is a point in $[a,b]$. What is the degree of precision of this rule? Justify your answer.

(c) The composite version of this rule is used on $[a,b]$ by dividing the interval into n subintervals. Let $h = (b-a)/n$. What is the order of convergence (i.e. m in $O(h^m)$) for the method? Prove your result.

(d) This composite version is applied to find an approximation to $I(g) = \int_2^3 g(x)dx$, first by dividing $[2,3]$ into 10 subintervals, then 20, then 40. It is observed that the error for these approximations is 0.1, 0.025, and 0.00625 respectively. What order of convergence does this represent? What does this suggest to you about the function g?

2. (a) Consider the following method for the ODE $y'(x) = f(x, y(x)),\ y(x_0) = Y_0$:

$$y_{n+1} = y_n + \frac{h}{12}(23f(x_n, y_n) - 16f(x_{n-1}, y_{n-1}) + 5f(x_{n-2}, y_{n-2})).$$

Here $x_n = x_0 + nh$ and $y_0 = Y_0$. Let $T_n(Y)$ be the truncation error. Use a theorem (or Taylor series approximations) to find the order m if $\frac{1}{h}T_n(Y)$ is $O(h^m)$.

(b) Suppose the initial conditions are approximated with enough accuracy. Will the method converge? If so, why, and at what order?

(c) The backward Euler method is given by $y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$. Give one advantage and one disadvantage it has over the usual Euler method.

3. Consider the following **non-standard** interpolation problem: given a C^1 function $f : [x_0, x_1] \to \mathbb{R}$, find a quadratic polynomial p so that

$$p(x_0) = f(x_0), \quad p'(x_0) = f'(x_0), \quad p'(x_1) = f'(x_1). \tag{1}$$

   (a) Show that the problem (1) has a unique solution.

   (b) Assuming $x_0 < x < x_1$, derive an error formula (a reasonable and correct upper bound) for $|f(x) - p(x)|$ under the assumption that the function f is sufficiently many times differentiable. If $x_1 - x_0 = h$, express the approximation order as $O(\|f^{[r]}\|_\infty h^p)$ with p an appropriate power, and r the appropriate derivative.

   Hint: Try to take advantage of the "more standard" relationship between p' and f'.

4. Halley's method for solving the nonlinear equation $f(x) = 0$ for a C^3 function f with $f'(x) > 0$ is based on applying Newton's method to the equation $F(x) = 0$ with

$$F(x) = \frac{f(x)}{\sqrt{f'(x)}}.$$

   It results in the iteration

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}. \tag{2}$$

   (a) Apply the iteration (2) to the equation

$$e^x = b$$

   to derive a fixed point iteration (i.e., an iteration of the form $x_{n+1} = g(x_n)$) that converges to $x^* = \ln b$. Simplify the expression of g in this iteration as much as possible. With $b = 5$ and $x_0 = 1$ compute (numerically) x_1 and x_2 (keep at least 5 significant digits). What is the actual error e_2 (it should be less than 10^{-5} if your iteration is correct)?

   (b) Explain briefly what is means for a fixed point iteration to be order-p convergent, and what is a practical way of proving order-p convergence. Does order-p convergence imply convergence for all initial values?

   (c) Show that your iteration at (a) converges with order 3 to $\ln b$. (It can be shown that Halley's method is of order 3 in general; if that's what you prefer, you can show the general case.)

   (d) Show that if $A \leq \ln b \leq B$, then $g([A, B]) \subseteq [A, B]$ and that g is contractive on $[A, B]$. Conclude that your iteration is globally convergent, i.e., for any $x_0 \in \mathbb{R}$, $x_n \to \ln b$.
   Hint: Show that $g'(x) \geq 0$ first.