Math 630 Comprehensive Examination

August, 2022

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Householder matrices (called equivalently reflectors) form one of the most important 'building blocks' in several numerical linear algebra methods.

   (a) Write down the general form (definition) of a Householder matrix H.

   (b) Show from this form that H is both Hermitian and unitary.

   (c) Given two vectors x and y, describe the two conditions on these vectors such that one can find a Householder matrix H satisfying $Hx = y$. Show that these conditions indeed are required.

   (d) Assuming the conditions in part (c) are satisfied, describe how one actually determines this matrix H when given x and y.

   (e) Describe how these Householder matrices can be used to similarity transform a square matrix to upper Hessenberg form.

2. For all parts of this problem, assume all matrices and vectors are real.

   (a) Write down the steepest descent method for solving $Ax = b$, where A is symmetric and positive definite.

   (b) Explain how the formulas from part (a) can break down if A is symmetric, but only non-negative definite (also called positive semi-definite).

   (c) Suppose that A is symmetric but only non-negative definite. Show that if b is in the range of A, then the steepest-descent method will still converge to a solution. You may assume that the steepest-descent method converges whenever A is symmetric and positive definite.

3. (a) Show that the following matrices $\hat{Q}$ and $\hat{R}$ specify a (condensed) QR-factorization of the given matrix A. Make sure to check all relevant properties of all matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} \\ 1 & 0 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

   (b) Derive, how a QR-factorization is used to solve the least squares problem given by $Ax = b$.

(c) Find the solution to the least squares problem $Ax = b$ with A from above and $b = [3, 1, -2]^T$.

4. (a) State the theorem describing the existence of a Cholesky factorization for a matrix.

   (b) Compute the Cholesky factorization of the following matrix (or show it does not exist):

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{bmatrix}.$$

   (c) Show that if R is the Cholesky factor of a matrix A, then

$$\|A\|_2 = \|R\|_2^2, \quad \text{and} \quad \kappa_2(A) = \kappa_2(R)^2,$$

   where $\kappa_2(A)$ denotes the condition number of A with respect to the 2-norm.