Math 620 Comprehensive Examination

August, 2022

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Consider the iteration $x_{n+1} = ax_n + b(1 - x_n^2)$.
   - (i) For what values of a and b will this iteration converge to $\alpha = 1$ when the initial guess x_0 is chosen close enough to 1?
   - (ii) Indicate the order of convergence for the cases above. Justify.

   (b) Consider Newton's Method for finding the solution α of $f(x) = 0$, where $f'(\alpha) \neq 0$.
   - (i) Use a theorem on fixed point iteration to show that this will give at least quadratic convergence to α when the initial guess is chosen close enough to α.
   - (ii) What will be the limit of $\frac{\alpha - x_{n+1}}{\alpha - x_n}$ and of $\frac{\alpha - x_{n+1}}{(\alpha - x_n)^2}$ as $n \to \infty$?

2. (a) Let $f(x)$ be in $C[a, b]$. Let $p_n(x)$ be the minimax approximation to f in the set P_n of polynomials of degree $\leq n$ and $q_n(x)$ be the best least squares approximation to f in the same set P_n (with weight function =1). Say whether or not

$$\|f - q_n\|_2 \leq \|f - p_n\|_2$$

   will always be true, giving a reason.

   (b) Suppose $\{\phi_0, \phi_1, \phi_2\}$ are orthogonal polynomials on $[0, 1]$ corresponding to some weight $w(x)$, such that for $i = 0, 1, 2$, we have

$$\int_0^1 w(x)(\phi_i(x))^2\, dx = i + 1.$$

   Also, suppose the function $f \in C[0, 1]$ is such that

$$\int_0^1 w(x)f(x)\phi_i(x)\, dx = 2i^2 + 1$$

   and

$$\|f\|_2 = \left(\int_0^1 w(x)(f(x))^2\, dx\right)^{\frac{1}{2}} = 7.$$

   Find $\|f - r_2^*\|_2$ where $r_2^* \in P_2$ is the best least squares quadratic approximation to f.

   (c) What values of a, b, c, d make $f(x)$, defined below, a cubic spline on $[-1, 1]$?

$$\begin{aligned} f(x) &= x^3, & x \in [-1, 0), \\ &= ax^3 + bx^2 + cx + d, & x \in [0, 1]. \end{aligned}$$

3. Derive the *local truncation error* $\tau_n(Y)$ of the numerical method

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}\Big(4f(x_{n+1}, y_{n+1}) - f(x_n, y_n) + 3f(x_{n-1}, y_{n-1})\Big), \quad n \geq 1,$$

for the ordinary differential equation $y'(x) = f(x, y(x))$. Be sure to obtain the precise coefficient of the leading order of h in the result that should read $\tau_n(Y) = C\, h^p\, Y^{(q)}(x_n) + \mathcal{O}(h^r)$ with some non-zero constant C and some integers p, q, and r, where $Y(x)$ denotes the solution to the differential equation. What is the order of accuracy of the method?

4. (a) Determine the weights A, B, C in order for the quadrature

$$Q(f) = Af(-h) + Bf(0) + Cf(h) \approx \int_0^h f(x)dx$$

to have the highest possible degree of precision. (Note that one of the nodes lies outside the interval where the function is integrated.)

(b) For the weights you identified, prove an estimate of the form

$$\left| Q(f) - \int_0^h f(x)dx \right| \leq Ch^p \max_{x\in[a,b]} |f^{[r]}(x)|,$$

where the numbers p, r as well as the interval $[a, b]$ need to be specified.