Math 620 Comprehensive Examination

January, 2022

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Describe (at least) three methods for the computation of an interpolating polynomial to a given dataset. Discuss their relative advantages, for instance, relative to computational cost or applicability or purpose of use. Hint: Start by clearly introducing notation for the problem to be solved.

   (b) Compute the interpolating polynomial to the data

0	4	2	-2
1	2	0	2

   Simplify your result to the form of a standard polynomial.

2. Consider the general problem of solving an equation of the form

$$f(x) = 0, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function, and assume x^* is a solution, that is, $f(x^*) = 0$.

   (a) Describe what it means for an iterative scheme for solving (1) to be of order r, with $r \geq 1$, and give examples (without many details) of iterative methods of various orders (not necessary integer orders).

   (b) Assume the equation (1) is transformed into an iteration of the form $x_{n+1} = g(x_n)$, with g being a twice continuously differentiable function that satisfies

$$g(x^*) = x^*, \quad g'(x^*) = 0, \quad g''(x^*) \neq 0.$$

   Show that the above method is exactly of order $r = 2$, and explain under what conditions on the initial value x_0 the iterates will converge to x^*.

   (c) Use the result at (b) (by identifying the corresponding function g) to show that Newton's method for solving (1) is of order $r = 2$ if $f'(x^*) \neq 0$, and of order $r = 1$ if $f'(x^*) = 0$ and $f''(x^*) \neq 0$.
   Hint: For the last statement you may use the fact that $f(x^*) = f'(x^*) = 0$ and $f''(x^*) \neq 0$ imply that there exists a sufficiently smooth function h so that $f(x) = (x - x^*)^2 h(x)$ with $h(x^*) \neq 0$.

3. In the following, $\mathcal{P}_n$ denotes the set of polynomials of degree $\leq n$.

   (a) Let $r_n^* \in \mathcal{P}_n$ be the best least squares approximation to f, a function that is continuous on $I = [-1, 1]$. Show that

   $$\int_I r_n^*(x)p(x)\,dx = \int_I f(x)p(x)\,dx$$

   for all $p \in \mathcal{P}_n$.

   (b) The three-point Gauss-quadrature rule on $I = [-1, 1]$ (with weight 1) is given by

   $$\int_I f(x)\,dx \approx \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}}).$$

   Find the corresponding rule on the interval $[0, 4]$ (i.e. give the weights and nodes).

   (c) Let $h = (b-a)/2$, and denote Simpson's rule for approximating $I(f) = \int_a^b f(x)\,dx$ by $I_h(f)$. We know this has the error estimate

   $$I(f) - I_h(f) = -\frac{h^5}{90}f^{(4)}(\eta)$$

   where η is some point in $[a, b]$. Use this to show that the rule's degree of precision is 3.

4. Consider the ordinary differential equation

   $$y'(x) = f(x, y(x)),\ \ y(x_0) = Y_0. \tag{2}$$

   (a) Replace the derivative term in (2) by two different numerical difference schemes to obtain two common one-step methods for solving ODEs - one explicit, and one implicit. (Each should have uniform step size h. You don't have to give the truncation error.)

   (b) Suppose $f(x, y) = \lambda y$ ($\lambda < 0$ is real) and $Y_0 = 1$. For each of your methods, derive a formula for the approximate solution y_n (i.e. the approximation to the exact solution at $x_n = nh$).

   (c) We know that the exact solution of the ODE in part (b) above tends to 0 as $x \to \infty$. Let h be fixed, and let $n \to \infty$ (so that $x_n \to \infty$). Under what conditions on h does $y_n \to 0$ for each of the two schemes?