Math 630 Comprehensive Examination

January, 2021

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. (a) Show that for all $x \in \mathbb{R}^n$

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2 \leq n\,\|x\|_\infty .$$

Hint: The inequality $\|x\|_2 \leq \|x\|_1$ becomes obvious on squaring both sides. The inequality $\|x\|_1 \leq \sqrt{n}\,\|x\|_2$ is obtained by applying the Cauchy-Schwarz inequality to the vectors x and $y = [1, 1, \ldots, 1]^T$.

(b) Make systematic use of the inequalities from part (a) to prove that for all $A \in \mathbb{R}^{n\times n}$

$$\|A\|_1 \leq \sqrt{n}\,\|A\|_2 \leq n\,\|A\|_1 .$$

2. Assume A is nonsingular, $\|\delta A\| / \|A\| < 1/\kappa(A)$, $b \neq 0$, $Ax = b$, and

$$(A + \delta A)(x + \delta x) = b + \delta b.$$

The symbol $\|\cdot\|$ denotes either a vector norm or a compatible matrix norm, respectively, depending to what object it is applied.

(a) Derive the equality

$$\delta x = A^{-1}(\delta b - \delta A(x + \delta x))$$

and prove the inequality

$$\frac{1}{\|A\|} \leq \frac{\|x\|}{\|b\|}.$$

(b) Use part (a) to show that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}.$$

3. (a) Show that the matrices $\hat{Q}$ and $\hat{R}$ specify a (condensed) QR-factorization of the given matrix A. Make sure to check all relevant properties of all matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} \\ 1 & 0 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \end{bmatrix}.$$

(b) Derive, how a QR-factorization is used to solve the least squares problem given by $Ax = b$.

(c) Find the solution to the least squares problem $Ax = b$ with A from above and $b = [3, 1, -2]^T$.

4. Let A be the non-symmetric matrix

$$A = \begin{bmatrix} 1 & \rho \\ -\rho & 1 \end{bmatrix}, \quad \rho > 0 \,,$$

and consider iterative methods for solving the system

$$Ax = b \tag{1}$$

with $b = [1, 2]^T$.

(a) With $x_0 = [0, 0]^T$ as initial guess, compute the iterates x_1 and x_2 for the Jacobi and the Gauss-Seidel iterations, respectively, for solving the system (1).

(b) Compute the error propagators for the Jacobi and the Gauss-Seidel methods, respectively, that is, the matrices $E = E_J$ or $E = E_{GS}$ for which the error vectors satisfy $e_{n+1} = Ee_n$.

(c) For each of the two methods determine all the values of ρ for which the method converges. Compare the convergence rates of the Jacobi and the Gauss-Seidel methods, using the spectral radii of the error propagators as proxies.