Math 620 Comprehensive Examination

January, 2021

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Consider the model of a binary computer number system, whose non-zero (and non-special) numbers are modeled by
$$\tilde{x} = (-1)^s \, (b_0.b_1 b_2 \dots b_{p-1})_2 \, 2^E$$
with sign bit $s \in \{0, 1\}$, binary digits $b_i \in \{0, 1\}$, $i = 0, 1, \dots, p-1$, and exponent $E_{\min} \leq E \leq E_{\max}$, where p, $E_{\min}$, $E_{\max}$ are the given parameters of the number system.

   (a) The normalized numbers $\tilde{x}$ are defined by $E_{\min} \leq E \leq E_{\max}$ and $b_0 \neq 0$ being the first non-zero digit of the number. Determine the formulas for the smallest and largest positive normalized numbers.

   (b) The de-normalized numbers $\tilde{x}$ are defined by $E = E_{\min}$ and the requirement that not all digits b_i, $i = 0, 1, \dots, p-1$, are zero. Determine the formulas for the smallest and largest positive de-normalized numbers.

   (c) Consider the very small number system with the given parameters $p = 3$, $E_{\min} = -1$, $E_{\max} = 2$. Compute and list **all** positive numbers of this number system (both normalized and de-normalized), and draw these positive numbers on a number line for $\tilde{x}$.

2. The bisection method for finding a solution α of $f(x) = 0$ starts with a "bracket" interval $[a, b] = [a_0, b_0]$ for which $f(a)f(b) < 0$. Let $x_0 = (a_0 + b_0)/2$ be the approximate solution at step 0, x_1 (the midpoint of the next bracket $[a_1, b_1]$) be the approximate solution at step 1, and so on. In the following, assume that there is only one exact solution $\alpha \in [a, b]$.

   (a) Given $\epsilon > 0$, how large should n be to guarantee that the error $|\alpha - x_n| \leq \epsilon$?

   (b) Suppose the method is programmed to stop if $f(x_n)$ is ever exactly 0, i.e. if the approximate solution x_n at step n turns out to be exactly equal to α. You run the program, and sure enough, it stops at step n. Give the set of all possible values that α can take (this will be a formula involving a, b and n).

   (c) Suppose you are in the situation above, i.e. the program has encountered the exact solution and stopped at step n. What is the value of $|x_n - x_{n-1}|$?

3. For $i = 0, 1, \ldots n$, let $x_i = -1 + ih$, where $h = 2/n$. These are just the evenly spaced points on the interval $[-1, 1]$. Denote by $\{l_i(x)\}, i = 0, 1, \ldots n$, the usual Lagrange interpolating polynomials on $\{x_i\}$, in the context of polynomial interpolation.

   (a) Show that the nth derivative of $l_n(x)$ equals h^{-n}. (Hint: What is the coefficient of x^n in the polynomial $l_n(x)$?)

   (b) Will the nth derivative of all the remaining $l_i(x)$ also equal h^{-n} or can these values be different? Justify briefly. (Hint: Do a rough calculation for $i \approx n/2$.)

   (c) Suppose $\{P_i(x)\}$ are the usual Legendre polynomials ($P_0(x) = 1, P_1(x) = x$, etc.). If n is even, give two values of $x \in [-1, 1]$ for which $P_{n+1}(x) = l_n(x)$. (Hint: Use the properties that $P_n(\pm 1) = (\pm 1)^n$ and also that P_n is an odd function for n odd and an even function for n even.)

4. Consider the weight function $w : [-1, 1] \to \mathbb{R},\ w(x) = x^2$.

   (i) Construct an integration rule of the form $I(f) = Af(-1) + Bf(0) + Cf(1)$ so that

$$I(f) = \int_{-1}^{1} w(x)\, f(x)\, dx$$

   is valid for all quadratic polynomials f.

   (ii) Determine the degree of precision of the quadrature at (i).

   (iii) Use the interpolation error formula to derive an estimate for the error

$$\left| \int_{-1}^{1} w(x) f(x) - I(f) \right| ,$$

   where f is a sufficiently smooth function (specify how many times does f have to be differentiable so that your estimate holds).