Math 630 Comprehensive Examination

August 18, 2020

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. Consider the 2×2 **square** matrix $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$. Show all steps of your calculations.

   (a) Compute on paper in exact arithmetic (that is, using fractions and square roots) the real SVD of A in the form $A = U\Sigma V^T$ with singular values σ_i, $i = 1, 2$, in Σ and orthogonal matrices U and V.

   (b) Compute the eigenvalues λ_i, $i = 1, 2$, of A in exact arithmetic (that is, using fractions and square roots).

   (c) Verify in exact arithmetic that $\det(A) = \lambda_1\lambda_2$ and $|\det(A)| = \sigma_1\sigma_2$.

2. Recall the definition of the Frobenius norm

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2 \right)^{1/2}$$

   that applies to any rectangular matrix $A = (A_{ij}) \in \mathbb{R}^{m\times n}$ with integers $m \geq 1$, $n \geq 1$ (including the case of A being a row or column vector).

   (a) Show the classical identity

$$\|A\|_F = \sqrt{\text{trace}(A^T A)}$$

   using the trace of a matrix, that is, the sum of its diagonal entries.

   (b) Show that with any orthogonal matrix $Q \in \mathbb{R}^{m\times m}$, the Frobenius norm satisfies the identity

$$\|QA\|_F = \|A\|_F.$$

   (c) If A is the special matrix given by the outer product $A = uv^T$ of two (column) vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, show the identity

$$\|A\|_F = \|u\|_F \|v\|_F.$$

   (Note the obvious notational identity $\|x\|_F = \|x\|_2$ for any row or column vector x.)

3. Consider building a basis of a Krylov subspace $\mathcal{K}_{k+1}(A, q) = \text{span}\left\{q, Aq, \ldots, A^k q\right\}$ using the Arnoldi process, where the first step entails normalization $q_1 = q/\|q\|_2$ and in the subsequent steps we take

$$\widetilde{q}_{k+1} = Aq_k - \sum_{j=1}^{k} q_j h_{jk}.$$

   (a) State why is the basis as stated above not ideal for numerical calculations? Explain briefly the idea of the algorithm, state to which other algorithm it is closely related and in particular define h_{jk} and $h_{k+1,k}$, so that $q_{k+1} = \widetilde{q}_{k+1}/h_{k+1,k}$.

   (b) Deduce that

$$Aq_k = \sum_{j=1}^{k+1} q_j h_{jk}, \qquad k = 1, 2, 3, \ldots.$$

   Consider the entries h_{jk} collected in a matrix H (respective submatrices H_m and $H_{m+1,m}$), and derive the useful identity

$$AQ_m = Q_m H_m + q_{m+1} h_{m+1,m} e_m^T,$$

   where e_m denotes the mth standard basis vector in $\mathbb{R}^m$.

(c) Consider the symmetric Lánczos process in, which the Hessenberg matrix H_m is typically denoted by T_m. Show that $T_m = Q_m^T A Q_m$, and T_m is symmetric if A is. Deduce that T_m is tridiagonal.

4. (a) State Generic Descend Algorithm, i.e., with unspecified search direction and consider exact line search, so that $\alpha = p^T r / p^T q$, where $q = Ap$.

   (b) Specify the search direction for Steepest Descend (SD) both with and without preconditioning. How are the search directions p_i and p_j related (in the geometric sense) in the conjugate gradient method (CG)? Would you prefer to use SD or CG in practice?

   (c) Apply one step of the steepest descend algorithm to the system

$$\begin{bmatrix} 10 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \end{bmatrix},$$

   starting with $x^{(0)} = 0$ using a) no preconditioner, b) the Jacobi preconditioner.