Math 630 Comprehensive Examination

August 22 (Day 2), 2019

**Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.**

1. (a) Let $M$ be any $n \times n$ nonsingular matrix, and let $A = M^T M$. Show that $A$ is symmetric, positive definite.

   (b) We say $A = R^T R$ is the Cholesky decomposition when $R$ has a special form. What is the form of $R$ and what can be said about its diagonal entries? In particular, under what conditions is the Cholesky decomposition unique?

   (c) Consider the *outer-product* form of Cholesky method, written in the form

$$\begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & t^T \\ s & \hat{R}^T \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ t & \hat{R} \end{bmatrix}.$$

   First, what is $t$? Use this form to formulate a recursive method for the computation of the Cholesky factorization, and then use it to find a factorization of the matrix

$$A = \begin{bmatrix} 9 & 3 & 3 \\ 3 & 10 & 7 \\ 3 & 5 & 9 \end{bmatrix}.$$

2. Let $A \in \mathbb{C}^{n\times n}$ and consider the eigenvalue problem $Au = \lambda u$.

   (a) Describe an iterative method to aproximate the eigenvector corresponding to the eigenvalue closest to some number $\rho_0 \in \mathbb{C}$ (which may not be an eigenvalue).

   (b) Let $q$ be a vector that is close to an eigenvector. Explain how to compute an approximation of the corresponding eigenvalue $\rho$. In particular, define the Rayleigh quotient which approximates $\rho$, and state how it is related to the value of $\|Aq - \rho q\|_2$.

   (c) Let $(\lambda, v)$ be an eigenpair of $A$, and assume $\|v\|_2 = 1$. Let $q \in \mathbb{C}^n$ with $\|q\|_2 = 1$, and $\rho = q^* A q$. Show that

$$|\lambda - \rho| \leq 2\,\|A\|_2\,\|v - q\|_2\,.$$

   Hint: You may use $\lambda = v^* A v$.

3. (a) Define the Householder reflector $R$ that transforms a unit vector ($\|u\|_2 = 1$) in $\mathbb{R}^n$ into the first standard basis vector $e_1 = [1, 0, \ldots, 0]^T \in \mathbb{R}^n$.

   (b) Briefly define the singular value decomposition (SVD) of a matrix $A \in \mathbb{R}^{m\times n}$ and the pseudo-inverse of $A$.

   (c) Let $u \in \mathbb{R}^n$, $n \geq 2$ be a unit vector. Let $A = [u\ 2u]$, that is, the $n \times 2$ matrix that has as columns the vectors $u$ and $2u$. Compute the SVD of $A$ and also the pseudo-inverse of $A$.
   Hint: For the left matrix in the SVD of $A$ use the matrix $R$ at (a); you may also exploit the relationship between the SVD of $A$ and the SVD of $RA$, which is easier to compute. The pseudo-inverse of $A$ has a very simple formula in terms of $A$.

4. (a) Define the condition number $\kappa(A)$ of a nonsingular square matrix $A$, and explain its relevance to the approximate solution of the linear system $Ax = b$. You may refer to the perturbed system $A(x + \delta x) = b + \delta b$.

   (b) Show that $\kappa(A) \geq 1$.

   (c) If $A$ is a nonsingular **triangular** $n \times n$ matrix, and the condition number is computed with respect to the $\infty$-norm, show that

$$\kappa(A) \geq \frac{\max_{i=1}^{n} |a_{ii}|}{\min_{i=1}^{n} |a_{ii}|}.$$