Math 620 Comprehensive Examination

August 21 (Day 1), 2019

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. A function $g(x)$ is called a *contraction* on the interval $[a,b]$ if $g([a,b]) \subset [a,b]$ and moreover, there exists $0 \leq k < 1$ such that $\forall x, y \in [a,b]$ we have $|g(x) - g(y)| \leq k|x-y|$.

   (a) Find $d > 0$ such that the function $g(x) = \cos x$ is a contraction on $[0.5-d, 0.5+d]$. Justify fully. Hint: The cosine of 1 radian is about 0.54.

   (b) Let $g(x), d$ be as above. Show that $g(x)$ cannot be a contraction on *every* interval $[0.5-c, 0.5+c] \subset [0.5-d, 0.5+d]$. Hint: Suppose it *was* a contraction on every interval. What would that imply?

   (c) Write down an iterative scheme of the form $x_{n+1} = G(x_n)$ that converges quadratically to a fixed point of $g(x) = \cos x$ for initial guess close enough to the fixed point.

2. (a) Suppose $\{P_n\}$ are the Legendre polynomials, with the L_2 norm of P_n given by $\sqrt{\frac{2}{2n+1}}$. Let $\{Q_n\}$ be orthonormal polynomials on $[-1,1]$ corresponding to the weight $w(x) = x^4$. Justifying each step of the calculation fully, compute the value of

   $$I = \int_{-1}^{1} (P_5(x) + 2x^2 Q_1(x) + x^2 Q_2(x))^2 \, dx.$$

   Hint: Use orthonormality/orthogonality properties of both $\{P_n\}$ and $\{Q_n\}$ — don't try to explicitly find Q_1, Q_2.

   (b) Let p_1 be the linear interpolant of a function f at the points $x_0, x_1 (x_0 < x_1)$. Then by the theorem on interpolation error, if $f \in C^{(2)}[x_0, x_1]$,

   $$f(x) - p_1(x) = (x-x_0)(x-x_1)\frac{f^{(2)}(\xi(x))}{2}, \ x_0 < x < x_1$$

   for some $\xi(x)$ between x_0 and x_1.

   Determine $\xi(x)$ explicitly in the case $f(x) = \frac{1}{x}, x_0 = 1, x_1 = 2$. (It will be a function of x.)

3. (a) Consider the quadrature rule

   $$\int_a^{a+h} f(x)dx = Af(a-h) + Bf(a) + Cf(a+h).$$

   Find A, B, C which maximize the degree of precision.
   Hint: First derive the rule for $a = 0$ and then use a change of variable.

(b) State this degree of precision and verify it is not any higher.

(c) Suppose g is a function whose 3rd divided differences are all the same. You are given that $g(-2) = 1, g(0) = 3, g(1) = 4, g(2) = 5$. Show, without finding an explicit formula for g, that

$$\int_{-2}^{0} g(x)dx = 4.$$

Hint: First express it as the difference of two integrals. Then use a well-known quadrature rule for each. This question is unconnected to parts (a) and (b) above, and the rule in part (a) isn't applicable.

4. Consider the ordinary differential equation

$$y'(x) = f(x, y(x)), \ \ y(x_0) = Y_0. \tag{1}$$

(a) Use numerical integration to derive the trapezoidal method for the above with uniform step size h. (You don't have to give the truncation error.)

(b) Given below is a multistep method for solving (1) (with uniform step size h):

$$y_{n+1} = 3y_n - 2y_{n-1} + h\left(\frac{1}{2}f(x_n, y_n) - \frac{3}{2}f(x_{n-1}, y_{n-1})\right) \tag{2}$$

What is the truncation error for (2)? Is the method consistent? Is it stable? Is it convergent? If convergent, determine the order of convergence.