Math 620 Comprehensive Examination

8/22/2018

Instructions: You must show all your work to receive full credit. Partial answers will only receive partial credit. Please choose 3 of the 4 problems to solve. Please indicate which 3 problems you would like graded.

1. This problem is concerned with solving the equation $f(x) = 0$ using Newton's method, assuming f is a smooth (C^∞) function.

   (a) Assume that $f(a) = 0$, $f'(a) \neq 0$, and $f''(a) = 0$. Show that Newton's method converges with order at least 3.
   (Hint: regard the Newton iteration as a fixed point iteration.)

   (b) Let $f(x) = x - x^3$. Write the Newton iteration for finding the roots of f and identify an open interval A around $a = 0$ so that for every initial guess $x_0 \in A$ the sequence x_n generated from Newton's method converges to 0. What is the order of convergence of Newton's method to $a = 0$, assuming $x_0 \in A$? Justify your answers.

2. Consider the following three **non-standard** interpolation problems:
   Given three numbers A, B, C and three (not necessarily distinct) nodes $x_1, x_2, x_3 \in \mathbb{R}$, find a quadratic polynomial $q(x)$ so that

$$\text{Problem 1: } q(x_1) = A, \quad q(x_2) = B, \quad q'(x_3) = C. \tag{1}$$

$$\text{Problem 2: } q(x_1) = A, \quad q'(x_2) = B, \quad q'(x_3) = C. \tag{2}$$

$$\text{Problem 3: } q(x_1) = A, \quad q''(x_2) = B, \quad q'(x_3) = C. \tag{3}$$

   (a) For Problem 1, what conditions if any are needed on the nodes to ensure a solution exists for every possible A, B, and C? Also, suppose this condition is violated, what conditions on A, B, C would still ensure a solution?

   (b) Repeat for Problem 2.

   (c) Repeat for Problem 3.

3. (a) f is a polynomial such that for any x_0, x_1, x_2, x_3, the divided difference $f[x_0, x_1, x_2, x_3]$ is the same constant. Also, $f(-1) = 2, f(1) = 3$. Justifying fully, find

$$\int_{-\sqrt{3}}^{\sqrt{3}} f(x)\, dx.$$

(b) Let D_h be a difference formula for the derivative. Identify the coefficients A, B and C in the formula $D_h f(x) = Af(x-2h) + Bf(x-h) + Cf(x)$ so that for any sufficiently smooth function $f(x)$, we have the error $f'(x) - D_h f(x)$ is as good as possible in terms of order of convergence. Find this order of convergence (i.e. p in the order h^p) and say how smooth the function$f$ has to be to attain it.

4. Consider the ordinary differential equation

$$y'(x) = f(x, y(x)), \; y(x_0) = Y_0. \tag{4}$$

(a) Use Taylor series to derive Euler's method for the above (with uniform step size h) along with the method's truncation error.

(b) Given below is a multistep method for solving (4) (with uniform step size h):

$$y_{n+1} = 2y_{n-1} - y_n + h\left(\frac{5}{2}f(x_n, y_n) + \frac{1}{2}f(x_{n-1}, y_{n-1})\right) \tag{5}$$

What is the truncation error for (5)? Is the method consistent? Is it stable? Is it convergent? If convergent, determine the order of convergence.