MATH 630 Numerical Analysis Comprehensive Exam – Jan 25, 2018

Attempt any 3 of the 4 questions below. Show your work.

1. Suppose an $m \times m$ matrix A is written in the block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is $n \times n$ and A_{22} is $(m-n) \times (m-n)$. Assume that the upper-left $k \times k$ block $A_{1:k,1:k}$ is nonsingular for each k with $1 \le k \le m$.

   (a) Verify the formula

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$$

   for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the *Schur complement* of A_{11} in A.

   (b) Suppose A_{21} is eliminated row by row by means of n steps of Gaussian elimination (resulting in partial LU factorization). Show that the bottom-right $(m-n)\times(m-n)$ block of the resulting matrix is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$. (Hint: Denote by L^{-1} the matrix used to eliminate A_{21}, that is let $A = LU$ be the partial factorization, and $A_{11} = L_{11}U_{11}$. Write the structure of L^{-1} using matrices $L_{11}, 0, I$ and a matrix X, and use it to show the claim. It may be useful to recall that LU factorization is computed by applying a sequence of transformations L_k to matrix A.)

2. Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

   (a) Determine a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V, and specifically list the singular values, left singular vectors, and right singular vectors of A.

   (b) What are the 1-, 2-, ∞- and Frobenius norms of A?

   (c) Find A^{-1} not directly, but via the SVD.

   (d) Find the eigenvalues of A.

   (e) Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.

3. Let

$$A = \begin{bmatrix} 1 & s \\ -s & 1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

   (a) Compute the error propagator for the Jacobi iteration, that is, the matrix E_J for which $e_{n+1} = E_J e_n$, where e_n is the error at the n^{th} iterate. Find necessary and sufficient conditions on s so that the Jacobi iteration for the system $Ax = b$ converges for any initial value x_0 and right-hand side $b \in \mathbb{R}^2$. (Hint: Set the condition that $E_J^n \to 0$ as $n \to \infty$.)

(b) Compute the error propagator E_{GS} for the Gauss-Seidel iteration, and show that the Gauss-Seidel iteration for the system $Ax = b$ converges (for any initial value x_0 and right-hand side $b \in \mathbb{R}^2$) under the same conditions on s as the Jacobi method.

(c) Given a (formal) parameter s for which both iterations converge, $b = [1, 1]^T$ and $x_0 = 0$, compute the convergence rates for each of the Jacobi and Gauss-Seidel iterations for the system $Ax = b$. The convergence rate with respect to a norm $\|\cdot\|$ is defined by $\lim_{n\to\infty} \|r_{n+1}\|/\|r_n\|$ (provided the limit exists), where $r_n = b - Ax_n$ is the residual.

4. The condition number of a square $n \times n$ nonsingular matrix A with respect to a matrix norm $\|\cdot\|$ is defined to be

$$\kappa(A) = \|A\| \cdot \|A^{-1}\|.$$

(a) Show that if $b, b' \in \mathbb{R}^n$ are non-zero, and $Ax = b$, $Ax' = b'$, then

$$\frac{\|x - x'\|}{\|x\|} \leq \kappa(A) \frac{\|b - b'\|}{\|b\|} ,$$

where $\|x\|$ denotes a vector norm that is compatible with the given matrix norm.

(b) If $A = QR$ is the QR factorization of A, show that $\kappa_2(A) = \kappa_2(R)$, where $\kappa_2(A)$ is the condition number of A computed using the 2-norm.

(c) Let $\varepsilon > 0$ be a small number, and

$$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2+\varepsilon \\ 2-\varepsilon \end{bmatrix}.$$

Using the infinity-norm, compute $\kappa(A)$. Then solve the system $Ax = b$ and find b' so that

$$\frac{\|x - x'\|_\infty}{\|x\|_\infty} \approx \kappa(A) \frac{\|b - b'\|_\infty}{\|b\|_\infty},$$

where $Ax' = b'$.