MASTER'S COMPREHENSIVE EXAM
Math 603 - MATRIX ANALYSIS
August 2022

Solve any three problems and show your work. Please indicate which problems you are submitting for grading. Each problem is worth 10 points.

Q1. Let that A be an $m \times n$ and B be an $n \times p$ matrix.

(a) Show that if $\operatorname{rank}(A) = n$, then $\operatorname{rank}(AB) = \operatorname{rank}(B)$ and $N(AB) = N(B)$.

(b) Show that if $\operatorname{rank}(B) = n$, then $\operatorname{rank}(AB) = \operatorname{rank}(A)$ and $R(AB) = R(A)$. Interpret the result in terms of composition of surjective linear maps.

Q2. Let $\mathcal{P}_2$ be the vector space of quadratic polynomials with real coefficients with basis $\mathcal{B}_1 = \{1, x, x^2\}$, and $x_0 < x_1 < x_2$ be three real numbers. We define the function $T : \mathcal{P}_2 \to \mathbb{R}^3$ by $T(p) = [p(x_0), p(x_1), p(x_2)]^T$.

(a) Show that the function T is linear, and compute the matrix $[T]_{\mathcal{B}_1,\mathcal{B}_2}$ where $\mathcal{B}_2$ is the standard basis in $\mathbb{R}^3$.

(b) Prove that $\dim(N(T)) = 0$.

(c) Use the result at (b) to show that for every three numbers a_0, a_1, a_2 there exists a quadratic polynomial q so that $q(x_i) = a_i$, $i = 0, 1, 2$.

Q3. Let X and Y be two complementary subspaces of $\mathbb{R}^n$, that is, $X + Y = \mathbb{R}^n$, $X \cap Y = \{0\}$. Let $\mathcal{B}_X = \{u_1, u_2, \ldots, u_k\}$ be a basis in X, and $\mathcal{B}_Y = \{u_{k+1}, \ldots, u_n\}$ be a basis in Y, and P the matrix representing the projection on X along Y (i.e., $P(x + y) = x$ if $x \in X, y \in Y$).

(a) Show, using the definition, that $\mathcal{B} = \mathcal{B}_X \cup \mathcal{B}_Y$ is a basis in $\mathbb{R}^n$.

(b) Show that P is similar to a diagonal matrix using a transformation involving the basis vectors in $\mathcal{B}_X$ and $\mathcal{B}_Y$.

(c) Show that $\operatorname{trace}(P) = \operatorname{rank}(P)$.

Q4. Let A be an $r \times r$ real valued matrix, and I be the $r \times r$ identity matrix.

(a) Prove that if the entries of A satisfy $\sum_{j=1}^{r} |A_{ij}| < 1$ for each $i = 1, 2, \ldots, r$, then $I + A$ is invertible.

(b) Prove that if A satisfies $A^T = -A$, then $\det(aI + A) \neq 0$ for all $a \in \mathbb{R} \setminus \{0\}$.

Q5. Let A be an $n \times n$ real matrix for which there is a vector $m \in \mathbb{R}^n$ with non-zero entries ($m_i \neq 0$ for all $i = 1, 2, \ldots n$) so that they satisfy

$$2|A_{ii}m_i| > \sum_{j=1}^{n} |A_{ij}m_j|, \ \forall i = 1, \ldots, n.$$

(a) Define the matrix B by $B_{ij} = A_{ij}m_j$, $1 \leq i, j \leq n$ and show that B is strictly diagonally dominant.

(b) Use your result at (a) to show that A is non-singular.