MASTER'S COMPREHENSIVE EXAM IN
Math 600–REAL ANALYSIS
August 2022

Do any three (out of the five) problems. Show all work. Each problem is worth ten points. In the following, $\mathbb{R}^n$ carries the usual metric.

Q1 Let (M, d) and (N, σ) be metric spaces and $f : (M, d) \to (N, \sigma)$ be continuous.

(i) Provide the definition of (open cover) compactness of a set K in (M, d). Show that if K is compact in M, then $f(K)$ is compact in N.

(ii) Provide the definition of a connected set C in (M, d). What are connected sets in $\mathbb{R}$? If C is connected in M, what can you say about $f(C)$ in N?

(iii) Suppose that (X, d) is a metric space such that finite sets are the only compact sets in X. Show that every continuous function from $[0, 1]$ (with the usual metric) into X is a constant.

Q2 (i) Let (M, d) be a complete metric space, and $S \subseteq M$ be a closed set. Show that (S, d) is complete.

(ii) Let $A \subseteq M$ be a compact set in (M, d). For each n, let $f_n : (M, d) \to \mathbb{R}$ be a continuous function with (f_n) converging to f_* uniformly on A.

(ii.a) Show that for any sequence (x_n) in A converging to x_*, $(f_n(x_n))$ converges to $f_*(x_*)$.

(ii.b) Explain why for each n, f_n restricted to A has a minimizer x_n^* in A.

(ii.c) Show that the sequence (x_n^*) has a subsequence that converges to a minimizer $\widehat{x}$ of f_* on A.

Q3 Let $C[0, 1]$ denote the set of continuous functions from the interval $[0, 1]$ into $\mathbb{R}$. State the Arzela-Ascoli theorem in the context of a sequence of functions (f_n) in $C[0, 1]$. Let $K \subseteq C[0, 1]$ be defined by the condition that $f \in K$ if and only if there exist $a, b \in \mathbb{R}$ and $m, n \in \mathbb{N}$ with $|a| \leq 1$ and $|b| \leq 1$ such that

$$f(x) = a\,\frac{\cos nx}{n} + b\,\frac{\sin mx}{m} \quad \forall x \in [0, 1],$$

where $\mathbb{N}$ denotes the set of all natural numbers. Show that

$$\min\left\{ \int_0^1 |\, t - f(t) \,|\, dt : f \in K \right\}$$

is attained on K.

Q4 (i) Discuss the uniform convergence of $\sum_{n=1}^{\infty} f_n$ on $[1, \infty)$ where $f_n := \frac{x^n}{n^2(1+x^n)}$.

(ii) Let $f(x)$ denote the sum of the above series for $x \in [1, \infty)$. Show that $f(x)$ is continuous on $[1, \infty)$ and that

$$f'(x) = \sum_{n=1}^{\infty} f_n'$$

for all $x \in (1, \infty)$. [Hint: For the derivative statement, consider $[\delta, \infty)$ for $\delta > 1$.]

Q5 (a) State the definition of Fréchet derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

(b) You are given $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{x^\alpha + y^\alpha}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0)$$

and $f(0, 0) = 0$ where $\alpha > 0$.

i. If $\alpha > 3$ prove that f is Fréchet differentiable at $(0, 0)$.
[Hint: $|x|^\alpha, |y|^\alpha \leq (x^2 + y^2)^{\frac{\alpha}{2}}$.]

ii. If $\alpha = 3$ prove that f has directional derivatives along all vectors $(a, b) \in \mathbb{R}^2$ at $(0, 0)$.

iii. Explain why f is not Fréchet differentiable at $(0, 0)$ when $\alpha = 3$.