MASTER'S COMPREHENSIVE EXAM IN
Math 603-MATRIX ANALYSIS
January 2020

Solve any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 Let $\{u_1, u_2, u_3, \ldots, u_p\}$ be a basis for the subspace X of the vector space V.

(a) Show that $\{u_1 + u_2, u_1 - u_2, u_3, \ldots, u_p\}$ is a basis for X.

(b) Suppose two nonzero vectors $v, w \in V$ are such that $\text{span}\{v, w\} \cap X = \{0\}$ and v is not a multiple of w. Let $Y = \text{span}\{u_1, u_2, v, w\}$. Find $\dim(Y)$, and prove your finding.

(c) Consider the subspace Y given in (b). Find a basis for $X \cap Y$, and prove your finding. Also determine $\dim(X + Y)$ (without proof).

Q2 (a) Consider the inner product on $\mathbb{R}^{n\times n}$ given by $\langle A, B\rangle := \text{trace}(A^T B)$ for $A, B \in \mathbb{R}^{n\times n}$. (i) Show that a symmetric matrix in $\mathbb{R}^{n\times n}$ is orthogonal to a skew-symmetric matrix in $\mathbb{R}^{n\times n}$. (ii) Show that for any $A \in \mathbb{R}^{n\times n}$, there exist $B, C \in \mathbb{R}^{n\times n}$ such that $A = B + C$ and $\langle B, C\rangle = 0$.

(b) Let $P \in \mathbb{R}^{n\times n}$ be the matrix representation of a projector onto the subspace X along the subspace Y in $\mathbb{R}^n$. Show that each eigenvalue of P is either 0 or 1, and determine if P is diagonalizable.

(c) Given a matrix $A \in \mathbb{R}^{m\times n}$, let S be the matrix representation of the orthogonal projector onto the range of A. Show that $A^T S = A^T$.

Q3 Let A and B be $m \times n$ and $n \times p$ matrices over $\mathbb{R}$, respectively.

(a) Prove that $\dim N(AB) \leq \dim N(A) + \dim N(B)$. (Hint: let $V = \{x \in \mathbb{R}^p : ABx = 0\}$, $W = \{y = Bx : x \in \mathbb{R}^p, Ay = 0\}$, and apply the rank plus nullity theorem for operator $T_B : x \in V \mapsto Bx \in W$.)

(b) Prove that $rank(A) + rank(B) \leq rank(AB) + n$.

Q4 Let $A \in \mathbb{R}^{n\times n}$.

(a) Define what it means for A to be diagonalizable, and show that if A has pairwise distinct eigenvalues, then A is diagonalizable. You may restrict your argument to $n = 3$.

(b) Show that if A is diagonalizable and $k \geq 2$ is integer, then there exists an $n \times n$ matrix B, perhaps with complex entries, so that $B^k = A$. What is the largest possible number of such matrices $B \in \mathbb{C}^{n\times n}$? (This number depends on n and k.)

(c) Find all matrices B so that $B^2 = A$ for

$$A = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}.$$

You may leave them in factored form.

Q5 (a) If $A \in \mathbb{R}^{m\times n}$ has rank r show that there exist <u>full-rank</u> matrices $B \in \mathbb{R}^{m\times r}$ and $C \in \mathbb{R}^{r\times n}$ so that $A = BC$.
(Hint: begin with the equality $A = GE_A$ with G nonsingular and E_A being the reduced row echelon form of A.)

(b) Conversely, if $A \in \mathbb{R}^{m\times n}$ and it can be written as $A = BC$ with $B \in \mathbb{R}^{m\times r}$ and $C \in \mathbb{R}^{r\times n}$ with B and C of <u>full-rank r</u>, then $\text{rank}(A) = r$.

(c) Show that with A, B, C as in (a), the matrix $B^T A C^T \in \mathbb{R}^{r\times r}$ is nonsingular.