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MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
August 2018

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 (a) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $\mathbb{R}^n$, and A be a closed set in $\mathbb{R}^n$. Show that the set $\{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in A\}$ is closed.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $A \subset \mathbb{R}^n$ such that $f(A)$ is bounded and $\{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in A\}$ is closed. Show that f is continuous on A.

(c) Let (x_n) be a sequence in a *complete* metric space (M, d) such that $\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$. Show that (x_n) converges in (M, d).

Q2 (a) Use the *open cover definition* to show that the union of finitely many compact sets is compact.

(b) Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous on $\mathbb{R}^n$. Show that the nonempty set $\{x \in \mathbb{R}^n \mid \|x\| \leq 5,\ g(x) \geq 1\}$ is compact, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

(c) Let the set $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + \frac{y^2}{16} = 1\}$, and the functions $p(x, y) = 6x + x^2 y$ and $q(x, y) = xy - y^3 - 4$ for $(x, y) \in \mathbb{R}^2$. Show that (i) B is connected; and (ii) there exists $z \in B$ such that $p(z) = q(z)$.

Q3 Consider the series

$$\sum_{n=1}^{\infty} f_n(x),$$

where $f_n : [0, \infty) \to \mathbb{R}$ is given by $f_n(x) = e^{-n} e^{x/n}$.

(a) Prove that the series converges uniformly on $[0, a]$ for each $a > 0$.

(b) Discuss the continuity and differentiability of the sum of the series on $[0, \infty)$.

(c) Prove that the series does not converge uniformly on $[0, \infty)$.

Q4 Let $C[0, 1]$ denote the space of all real valued continuous functions on $[0, 1]$ endowed with the supremum norm metric.

(a) State the definition of 'equicontinuity' for a set in $C[0, 1]$. Is a singleton set in $C[0, 1]$ equicontinuous? Justify your answer.

(b) State a sufficient condition for a set E in $C[0, 1]$ to have the following property: Every sequence in E has a subsequence that converges in $C[0, 1]$.

(c) Show that in the set

$$E = \{f \text{ differentiable } : |f(0)| \leq 2,\ |f'(x)| \leq 3\ \forall x \in [0, 1]\},$$

every sequence has a subsequence that converges in $C[0, 1]$.

Q5 Provide the definition of the Fréchet derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

For $k > 0$, a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be homogeneous of degree k if for all $\alpha > 0$ and all $x \in \mathbb{R}^n$

$$f(\alpha x) = \alpha^k f(x).$$

(a) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is homogeneous of degree $k > 1$, then f has directional derivatives at the origin along v for all $v \in \mathbb{R}^n$.

(b) Prove that if f is homogeneous of degree $k > 0$ and is bounded on the closed unit ball $\overline{B}(0, 1)$ centered at the origin, then f is continuous at the origin.

(c) In addition to the previous boundedness condition, if f is homogeneous of degree $k > 1$ then prove that f is Fréchet differentiable at the origin and find the derivative.