MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
January 2019

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 (a) Let (M,d) be a complete metric space and A be a nonempty subset of M. Show that (A,d) is complete if and only if A is closed in (M,d).

(b) Let (x_k) be a convergent sequence in the metric space (M,d). Show that there exists a subsequence (x_{k_j}) such that $\sum_{j=1}^{\infty} d(x_{k_j}, x_{k_{j+1}}) < \infty$.

(c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and (x_k) be a *bounded* sequence in $\mathbb{R}^n$ such that $f(x_{k+1}) \geq f(x_k)$ for all k. Show that there exists $x_* \in \mathbb{R}^n$ such that $(f(x_k))$ converges to $f(x_*)$.

Q2 (a) Let (M,d) be a *discrete* metric space, i.e., for $x, y \in M$, $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$; let A be an infinite subset of M. Use the open cover definition to show that A is *not* compact. $\ast$

(b) Let A be a compact set in a metric space (M,d) and $f : (M,d) \to \mathbb{R}$ be continuous. Let S be the set of maximizer(s) of f on A, i.e., $S = \{x \in A \mid f(x) \geq f(z), \forall z \in A\}$. Show that S is nonempty and compact. $\ast$

(c) Consider a nonempty connected set C in $\mathbb{R}^n$ and let $E := \{x_1 \mid (x_1, \ldots, x_n) \in C\}$. Show that there is *no* continuous function from E *onto* $\{-1, 0, 1\}$.

Q3 (a) State the Banach contraction principle.

(b) On $\mathbb{R}$, consider the function $f(x) := 1 + \frac{1}{2}|x| + \frac{1}{8}\sin 3x$. Show that f has a unique fixed point in $\mathbb{R}$.

(c) Let f be as above and h be a Lipschitz function on $\mathbb{R}$. Show that there is an $\varepsilon > 0$ such that $f + \varepsilon h$ also has a unique fixed point in $\mathbb{R}$.

Q4 Let $C[0,1]$ denote the space of all real valued continuous functions on the interval $[0,1]$ with norm $||f|| := \max_{t\in[0,1]} |f(t)|$ and metric $D(f,g) := ||f - g||$. Let

$$K := \{f \in C[0,1] : ||f|| \leq 1\}$$

denote the closed unit ball in $C[0,1]$. For any natural number n, define the number $\alpha_n := (2n-1)\frac{\pi}{2}$ and functions f_n and g_n in $C[0,1]$ by

$$f_n(x) := x^n \sin(\alpha_n x), \text{ and } g_n(x) := \int_0^x f_n(t)dt.$$

(a) Define 'uniform convergence' of a sequence of continuous functions over $[0,1]$. How is it related to 'convergence in the D-metric'?

(b) Does the sequence f_n converge uniformly over $[0,1]$? Explain.

(c) Is the set K compact in the metric space $\big(C[0,1], D\big)$? Explain.

(d) Using Arźela-Ascoli theorem, show that g_n has a subsequence that converges uniformly over $[0,1]$.

Q5 (a) Given two finite dimensional normed vector spaces V and W, a map $f : V \to W$, and $c \in V$, provide the definition of the Fréchet derivative $Df(c)$ of f at c.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$f(x) = (a^T x - 1)x$$

where $a \in \mathbb{R}^n$ is non-zero, x, a are interpreted as column vectors and a^T denotes the transpose of a.

Show directly from the definition that the Fréchet derivative of f at $c \in \mathbb{R}^n$ is given by

$$Df(c)(v) = (a^T v)c + (a^T c - 1)v, \quad v \in \mathbb{R}^n.$$

Show that the derivative $Df(c)$ is non-zero at all $c \in \mathbb{R}^n$ if $n \geq 2$ and that the derivative is zero for a certain value of c when $n = 1$.