COMPREHENSIVE EXAMINATION

Math 650 / Optimization / January 2003

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Name ____

INSTRUCTIONS:
Solve any *two* out of the following three problems. You must *mark* clearly which two problems you would like to be graded. Each problem carries equal weight.

1. Consider the optimization problem

$$\begin{array}{rl} \max & x^2 + y^2 \\ s.t. & xy \leq 1 \\ & x \leq 2 \\ & y \geq 0 \end{array}$$

a. Sketch the feasible region.

b. Write the Fritz John conditions for the critical point(s) of the problem.

c. Show that every Fritz John point must satisfy the KKT conditions (that is, $\lambda_0 > 0$).

d. Determine which of the following three points satisfy the KKT conditions: **A.** $(0,0)$, **B.** $(2,0)$, **C.** $(2,1/2)$.

e. Among the three points in **d**, determine which ones are local maximizers. Use second order (necessary and sufficient) conditions for this purpose.

2. If C is a convex set, we define its *polar* as the set C°

$$C^\circ = \{y : y^T x \leq 1,\ \forall x \in C\}.$$

This problem is concerned with the determination of the polar of a polytope $P = \{x : Ax \leq b\}$ where A is an $m \times n$ matrix.

a. State the affine version of **Farkas Lemma** and use it to give an explicit description of P°.

b. Let $S = \{v_1, \ldots, v_k\}$ be the vertices of P (so that P is be the convex hull of S. Show that $P^\circ = \{y : v_i^T y \leq 1, i = 1, \ldots, k\}$.

Now consider the triangle $T = \{(x_1, x_2) : x_1 + x_2 \leq 1, x_2 \leq x_1 + 1, x_2 \geq -1\}$.

c. Determine T° explicitly using **a.**.

d. Determine T° explicitly using **b.**, and show that T° is a triangle. Find the vertices of T.

3. The purpose of this problem is to work out a dual program (D) to the convex minimization problem **(P)**: $\min_{x \in C} f(x)$, where $f(x) = \max\{f_1(x), \ldots, f_k(x)\}$, that is where the function $f(x)$ is the pointwise maximum of the convex functions $\{f_1(x), \ldots, f_k(x)\}$. Assume for simplicity that $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, k)$ and that C is a compact, convex set.

a. Prove the equality

$$\max\{u_1, \ldots, u_k\} = \max_{\lambda \in S^k} \lambda^T u,$$

where $S^k = \{\lambda \in \mathbb{R}^k : \lambda \geq 0, \sum_{i=1}^k \lambda_i = 1\}$ is the unit simplex in $\mathbb{R}^k$. Use it to write **(P)** as a min–max problem, and write the dual problem **(D)** as a max–min problem.

Now observe that we can write **(P)** as a *constrained* convex minimization problem

$$\min\{y : f_1(x) \leq y, \ldots, f_k(x) \leq y\}, \quad (P')$$

in the variables x and y.

b. Formulate the Lagrangian dual program **(D′)** of **(P′)**, and prove that *Strong Duality Theorem* holds true between **(P′)** and **(D′)**. (You may use any results proved in the course as long as you state the result correctly, and in full.)

c. Show that simplifying **(D′)** gives **(D)**, thereby proving that *Strong Duality Theorem* holds true between **(P)** and **(D)**.