COMPREHENSIVE EXAMINATION
Math 650 – Optimization
January 1999

INSTRUCTIONS:
You must do either Question 1 or Question 2 (25 points); Question 3 or 4 (10 pts.); Question 5 (25 pts.); Question 6 or 7 (25 pts.); and Question 8 (15 pts.). The exam is worth 100 points.
You must show all your work for full credit!

Q1. Answer fully *two* of the following three parts.

(a) State and prove Jensen's inequality for a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

(b) Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and denote by $\pi_C(x)$ the projection of x onto C. (That is, $\pi_C(x)$ is the unique solution to the problem $\min\{||z-x||^2 : z \in C\}$.) State the *variational inequality* for π_C and use it to prove that π_C is non–expansive, that is,

$$||\pi_C(x) - \pi_C(y)|| \leq ||x - y||, \qquad x, y \in \mathbb{R}^n.$$

(c) Let $P_1, P_2 \subseteq \mathbb{R}^n$ be two polytopes (bounded polyhedra). Prove that the Minkowski sum $P_1 + P_2$ is also a polytope. (*Hint:* use the fact that P_i is the convex hull of a finitely many points, $i = 1, 2$.)

Q2. Let $f : I = [a, b] \to \mathbb{R}$ be a convex function. If $x_1 < x_2 < x_3$ are points in I, then the following is a well known inequality:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}. \qquad (1)$$

(Draw a picture.)

(a) Prove the first inequality above.

(b) Use (1) to prove that the one–sided derivatives $f'_+(x), f'_-(x)$ exist for a point $x \in (a, b)$, where

$$f'_+(x) = \lim_{t \downarrow 0} \frac{f(x+t) - f(x)}{t}, \qquad f'_-(x) = \lim_{t \downarrow 0} \frac{f(x) - f(x-t)}{t}.$$

Q3. Let A be an $m \times n$ matrix. We proved in the course that the polyhedron $P := \{x : Ax \leq b\}$ has the useful representation

$$P = \{\sum_{i=1}^{k} \lambda_i v_i + \sum_{j=1}^{l} \delta_j d_j : \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0, \delta_j \geq 0\}.$$

Suppose that the linear function $c^T x$ is bounded from below on P. Use the representation of P above to prove that the linear program $\min\{c^T x : x \in P\}$ has an optimal solution which occurs at some vertex v_i of P.

Q4. Solve the problem

$$\max\{\prod_{i=1}^{n} x_i : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, i = 1, \ldots, n\}.$$

Q5. Consider the optimization problem

$$\begin{array}{rl} \min & x \\ \text{subject to} & (x+1)^2 + y^2 \geq 1 \\ & x^2 + y^2 \leq 2. \end{array}$$

(a) Which of the points $A = (0,0)$, $B = (-1,-1)$, and $C = (0,\sqrt{2})$ satisfy the *first* order necessary conditions (i.e. KKT conditions) for optimality?

(b) Which of the points A, B, C in (a) satisfy the *second* order necessary/sufficient conditions for optimality?

Q6. Consider the optimization problem

$$\begin{array}{rl} \min & (x-a)^2 + (y-b)^2 + xy \\ \text{subject to} & 0 \leq x \leq 1, \\ & 0 \leq y \leq 1, \end{array} \qquad (P)$$

where $a, b \in \mathbb{R}$ are parameters of the problem.

(a) Show that the problem (P) is a convex programming problem for all values of the parameters $a, b \in \mathbb{R}$.

(b) Write down the Karush–Kuhn–Tucker (KKT) conditions for (P), using the Lagrange multipliers λ_i for the ith constraint, $i = 1, 2, 3, 4$.

(c) For what values of the parameters a, b is the point $(x^*, y^*) = (1,1)$ optimal solution to (P)?

(d) Repeat (c) for the point $(x^*, y^*) = (0,1)$.

Q7. Consider the convex optimization problem

$$\begin{aligned} \min \quad & x^2 + y^2 \\ \text{subject to} \quad & x^2 - y \le 10 \\ & -x + y \le -4. \end{aligned}$$

(a) Sketch the feasible region.

(b) Show, either by appealing to a theoretical result (state carefully the result!) or by performing a calculation, that every point which satisfies the Fritz John conditions also satisfies the KKT conditions.

(c) Argue that there exists a unique optimal point which is also the unique KKT point. Compute this point using the KKT conditions.

Q8. Determine (calculate explicitly) the dual program to the program in Question 6 *or* in Question 7. State the *strong duality theorem* and explain whether it holds true for the problem at hand.

Q9. (Extra Credit, 5 pts.) Formulate the following min–max problem as a linear program:

$$\min_{x \in \mathbb{R}^n} \max_{1 \le i \le m} a_i^T x.$$