MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
August 2014

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 (a) Provide the sequential criterion for compactness of a set in a metric space.

(b) Let A be a subset of a metric space (M, d). Prove that closure of A is compact if and only if for every sequence (x_n) in A there exists a convergent subsequence (that converges in M).

HINT: Given a sequence in the closure of A construct an appropriate sequence in A.

Q2 (a) State the Arzela-Ascoli theorem in the context of functions $f : [0, 1] \to \mathbb{R}$.

(b) Let $C([0, 1])$ denote the space of continuous functions $f : [0, 1] \to \mathbb{R}$ equipped with the supremum norm. Prove that the closure of an equicontinuous subset of $C([0, 1])$ is equicontinuous.

(c) Define the map $I : C([0, 1]) \to C([0, 1])$ by

$$I(f)(x) = \int_0^x f(t)dt, \quad \forall x \in [0, 1],$$

for all $f \in C([0, 1])$. Suppose $K \subset C([0, 1])$ is bounded. Prove that the closure of $I(K)$ is compact. Note that $I(K)$ is the image of K under I:

$$I(K) = \{g \in C([0, 1]) \mid g = I(f) \text{ for some } f \in C([0, 1])\}$$

Q3 (a) State the Weierstrass approximation theorem on an interval $[a, b]$.

(b) For a continuous real valued function f on $[a, b]$ (in which case we write $f \in C[a, b]$), the moment sequence $(f_0, f_1, f_2, \ldots)$, is defined by $f_n := \int_a^b f(x)x^n \, dx$, $n = 0, 1, 2, \ldots$. If f and g in $C[a, b]$ have identical moment sequences, show that f and g are equal.

(c) Show that if $f \in C[a, b]$ with $f_n = 0$ for all $n \geq 5$, then f is identically zero.

Q4 Consider the series $\sum_{n=1}^{\infty} f_n(x)$ on $[0, \infty)$ where

$$f_n(x) = \frac{xe^{-n/x}}{x + n}, \; x > 0, \quad f_n(0) = 0.$$

(a) Show that the series converges uniformly on $[0, M]$ for every $M > 0$.

(b) Discuss the continuity of the sum on $[0, \infty)$.

(c) Show that the series does not converge uniformly on $[0, \infty)$.

Q5 (a) Provide the definition of the derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are normed vector spaces (possibly infinite dimensional).

(b) Let $C([0,1])$ be the space of continuous real valued functions on $[0,1]$ endowed with the supremum norm. Define $F : C([0,1]) \to C([0,1])$ by

$$F(f)(x) = \int_0^x e^{-2t}(f(t))^2 dt - \int_0^x e^{-t} f(t) dt,$$

for all $f \in C([0,1])$. Show directly from the definition that the derivative of F is differentiable on the entire domain.

(c) For the F defined above, show that $DF(f) = 0$ if and only if f is given by $f(x) = e^x/2$ for $x \in [0,1]$.