PhD COMPREHENSIVE EXAM IN
ORDINARY DIFFERENTIAL EQUATIONS
August 2012

Do any 3 of the following 4 problems. Show all work. Each problem is worth ten points.

Q1. (a) Provide the definition of what it means for two flows ϕ^t and ψ^t on $\mathbb{R}^n$ to be topologically conjugate.

(b) Suppose $\bar{x} \in \mathbb{R}^n$ is a stable equilibrium of flow ϕ^t on $\mathbb{R}^n$, and that flow ψ^t on $\mathbb{R}^n$ is topologically conjugate to ϕ^t via a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$. Prove that $h(\bar{x})$ is a stable equilibrium of ψ^t.

(c) Prove that the flow of $\dot{x} = -x$ and that of $\dot{x} = -2x$ (both on $\mathbb{R}$) are topologically conjugate.

Q2. Consider the two dimensional autonomous system $\dot{x} = f(x)$ where f is defined and continuously differentiable on all of $\mathbb{R}^2$, and $f(0) = 0$. Suppose the origin is (Lyapunov) stable. Prove that if $x(t)$ is a solution which has 0 in its ω-limit set then $\lim_{t\to\infty} x(t) \to 0$. *Note:* The '0' in these statements is a shorthand for the origin, that is $(0, 0)$.

Q3. You are given the system:

$$\dot{x}_1 = \sin(x_2) - x_1,$$
$$\dot{x}_2 = -\sin(x_1) - x_2.$$

(a) Decide the stability (unstable, stable, asymptotically stable) of the equilibrium $(0, 0)$.

(b) Explain why a unique solution exists for all $t \in \mathbb{R}$ for each initial condition $x(0) = (x_1(0), x_2(0)) \in \mathbb{R}^2$.

(c) Show that for all $t \in \mathbb{R}$ the following hold:

$$x_1(0)e^{-t} - 1 + e^{-t} \le x_1(t) \le x_1(0)e^{-t} + 1 - e^{-t},$$
$$x_2(0)e^{-t} - 1 + e^{-t} \le x_2(t) \le x_2(0)e^{-t} + 1 - e^{-t}.$$

Hint: Consider $t \ge 0$ and $t < 0$ separately. Use techniques similar to the proof of Gronwall Lemma.

(d) Prove that $A = [-1, 1]\times[-1, 1]$ is forward invariant and that for every initial condition $x(0)$ the corresponding solution $x(t)$ approaches A as $t \to \infty$.

Q4. (a) Find a Lyapunov function of the form $ax^2 + by^2$ to determine the stability of the origin of

$$\dot{x} = -\frac{1}{2}x^3 + 2xy^2,$$
$$\dot{y} = -y^3.$$

(b) Based on your answer is the origin asymptotically stable?