Comprehensive Examination

Statistics 653 - Basic Probability

100 points

August 23 , 2022 (9:30 am–1:00 pm)

Name: ________________________________

- You **can use only** calculator, but **not cell phone**
- You **must** show all proof details/calculations which lead to the answer

Q1. [12 points]

$X_1, \cdots, X_n$ are iid with $f(x|\theta) = \theta x^{\theta-1}$ $0 < x < 1$ and $\theta > 0$. Derive the MLE $\hat{\theta}$ of θ and compute its asymptotic variance. Does it attain the RCLB? If not, compute its efficiency.

Q2. [12 points]

$X_1, \cdots, X_n$ are iid $U\left[\theta - \frac{1}{2},\, \theta + \frac{1}{2}\right]$.

(a) Prove that the MLE of θ is *not* unique.

(b) Suggest an MLE of θ which is also unbiased and compute its variance.

Q3. [12 points]

A box contains N balls marked $1, \cdots, N$. In order to test $H_0 : N = M$ versus $H_1 : N < M$, n balls are randomly drawn from the box, and a test procedure rejects H_0 if $X_{(n)}$, the maximum number observed, is $\leq K$. Show that the power of this test is increasing as N moves away from M. Consider two cases: with and without replacement.

Q4. [12 points]

The pdf of a random vector X has the form $f(x|\theta) = C(\theta)h(x)e^{\theta T(x)}$ for x in R^p.

(a) Show that for a monotone increasing function $\psi(T)$ in T, $E[\psi(T)|\theta]$ is monotone increasing in θ.

(b) Use (a) to derive a UMP test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Describe the test procedure completely and show that its power is increasing in θ.

Q5. [16 points] $X_1, \cdots, X_n$ are iid $N(\theta, 1)$. Define $\psi(\theta) = P\left[N\left(\theta, \frac{1}{k}\right) > c\right]$ for some $c > 0$ and $1 < k < n$. Derive the UMVUE of $\psi(\theta)$ and compute its efficiency compared to RCLB.

Q6. [12 points]

(a) Prove that a statistic $T(X)$ is UMVUE of some θ if and only if $T(X)$ is uncorrelated with every unbiased estimate of 0.

(b) Hence show that if $T_i(X)$ is UMVUE of $g_i(\theta)$, $i = 1, \cdots, k$, then $\sum_{i=1}^{k} c_i T_i(X)$ is UMVUE of $\sum_{i=1}^{k} c_i g_i(\theta)$.

Q7. [12 points]

$X \sim B(n, \theta)$ and $\theta \sim Beta(\alpha, \beta)$. Derive the Bayes estimate of $\phi(\theta) = \theta^r(1-\theta)^s$ under squared error loss. For $r = s = 1$, compute its mean. Can it be unbiased for this case?

Q8. [12 points]

$X_1, \cdots, X_{2n+1}$ are iid according to $f(x) = \frac{1}{2}e^{-|x-\theta|}$, both x and θ being real. Derive the MLE of θ and its distribution. Prove that the MLE is an unbiased estimate of θ.