Comprehensive Examination

Statistics 651 - Basic Probability

100 points

August 22, 2022 (9:30 am–1:00 pm)

Name: ________________________________

- You **can use only** calculator, but **not cell phone**
- You **must** show all proof details/calculations which lead to the answer

Q1. [12 points]

There are only two kinds of buses at your place, Number 1 and 2. Arrivals of the Number 1 bus form a Poisson distribution of rate 1 bus per hour, and arrivals of the Number 2 bus form an independent Poisson distribution of rate 7 buses per hour. What is the probability that exactly 4 buses pass by in 1 hour ?

Q2. [12 points]

A coin with probability 2/3 of Heads is flipped repeatedly. The sequence of outcomes can be divided into runs (blocks of H's or blocks of T's) $\big[$for example, if the first 13 outcomes are HHHTTTTHTTTHH....., then it becomes HHH, TTTT, H, TTT, HH, which has 5 runs, with lengths 3, 4, 1, 3, 2, respectively$\big]$. Find the expected length of the first run.

Q3. [12 points]

Twelve players on a basketball team consist of 3 centers, 4 forwards, and 5 backcourt players. The players are paired up at random into four groups of size 3 each (triplet). Find the variance value of the number of triplets consisting of one of each type of player.

Q4. [8+8 points]

Suppose that $X_1, X_2, \ldots, X_n$ are independent exponential random variables with rate parameter 1.

(a) Prove that for any $t \in \mathbb{R}$

$$\lim_{n\to\infty} P\left(X_{(n)} - \ln(n) \leq t\right) = e^{-e^{-t}}.$$

(b) Suppose that for any $t \in \mathbb{R}$, $P(Y \leq t) = e^{-e^{-t}}$. Find a close-form expression of $E(Y)$ (you can use part (a) if needed).

Q5. [12 points]

For random variables X and Y with the finite $Var(X+Y), Var(X), Var(Y)$, show that

$$\sqrt{Var(X+Y)} \leq \sqrt{Var(X)} + \sqrt{Var(Y)}.$$

Q6. [12 points]

Let U and V be independent variables with the $\chi^2_{(r)}$ and $\chi^2_{(s)}$ distributions respectively. Set $F = \dfrac{U/r}{V/s}$. Compute $E(F)$. What are conditions on r and s such that $E(F)$ is finite?

Q7. [12 points]

Suppose that $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent random variables. Define $U = Y + X$ and $V = Y - X$. Find $P(V^4 < U^4)$.

Q8. [12 points]

Let $X_{(i)}$, $i = 1, \ldots, n$, denote the order statistics from a set of n continuous uniform $[0,1]$ random variables. Which value of i minimizes, and which value maximizes, $Var(X_{(i)})$?