MASTER'S COMPREHENSIVE EXAM IN
Math 600-REAL ANALYSIS
January 2018

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 Let $A \in \mathbb{R}^{m\times n}$ be a matrix. For any sets $\mathcal{S}, \mathcal{U} \subseteq \mathbb{R}^n$, define $A\mathcal{S} := \{Ax \mid x \in \mathcal{S}\}$ and $\mathcal{S}+\mathcal{U} := \{x+y \mid x \in \mathcal{S}, y \in \mathcal{U}\}$. Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$, and $\mathcal{C}$ be a compact set in $\mathbb{R}^n$. You may use the facts that a subspace of the Euclidean space is closed and that a linear mapping is continuous without proof.

(a) Show that $A\mathcal{V}$ is a closed set and $A\mathcal{C}$ is a compact set.

(b) Use (a) to show that $A\mathcal{V} + A\mathcal{C}$ is closed.

(c) Suppose $\mathcal{C}$ is path connected. Show that $A\mathcal{V} + A\mathcal{C}$ is path connected.

Q2 Let (f_n) be a sequence of real-valued functions that converges uniformly to f_* on a set $A \subseteq \mathbb{R}$, where each f_n is bounded on A, i.e., there exists $M_n > 0$ such that $|f_n(x)| \leq M_n$ for all $x \in A$.

(a) Show that f_* is bounded on A.

(b) Show that there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in A$ and all n.

(c) Let (a_n) be a bounded real sequence, and denote by $a_n f_n$ the product of a_n and f_n. Show that $(a_n f_n)$ has a subsequence which converges uniformly on A.

Q3 Let $f_n : \mathbb{R} \to \mathbb{R}$ be $f_n(x) = \dfrac{x}{n^r + x^2}$ with $r > 1$, and consider the series $s_* := \sum_{n=1}^{\infty} f_n$.

(a) Let $1 < r \leq 2$. Show that the series converges uniformly on any bounded set $A \subseteq \mathbb{R}$ and that s_* is continuous at any point in $\mathbb{R}$.

(b) Let $r > 2$. Show that the series converges uniformly on $\mathbb{R}$.

(c) Let $1 < r \leq 2$. Show that the series does not converge uniformly on $\mathbb{R}$.

Q4 (a) State the definition of a contraction map and state the contraction mapping theorem (also known as the Banach fixed point theorem).

(b) Let (M, d) be a compact metric space and for each $n \in \mathbb{N}$ let $f_n : M \to M$ be a contraction mapping. Further suppose that $f : M \to M$ and that (f_n) converges uniformly (on M) to f. Prove that f has a fixed point.

(c) Show via a counter example that in the previous question f need not have a unique fixed point. [HINT: Consider $M = [0, 1]$].

Q5 (a) Provide the definition of the Frechet derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are finite dimensional normed vector spaces.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = x_1x_2 \quad \text{if } x_1x_2 \in \mathbb{Q}$$
$$f(x) = 0 \quad \text{if } x_1x_2 \notin \mathbb{Q}.$$

Decide if f has a directional derivative along all $v \in \mathbb{R}^2$.
Decide if f is Frechet differentiable at $(0, 0)$.