MASTER'S COMPREHENSIVE EXAM IN
Math 600 -REAL ANALYSIS
August 2019

Do any three (out of the five) problems. Show all work. Each problem is worth ten points.

Q1 In $\mathbb{R}^n$ (with the usual metric), for elements $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, let $x*y = (x_1y_1, x_2y_2, \ldots, x_ny_n)$. Also, for sets A and B in $\mathbb{R}^n$, let $A*B := \{x*y : x \in A, y \in B\}$. Define functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) := x * x$ and $g(x, y) := x * y$. Prove the following statements:

(a) f and g are continuous.

(b) If A and B are two compact sets in $\mathbb{R}^n$, then so is the set $A * B$.

(c) If A and B are two connected sets in $\mathbb{R}^n$, then so is the set $A * B$.

Q2 State the *contraction mapping theorem* (also known as the *Banach fixed-point theorem*) by explicitly defining the term '(strict) contraction'.

Suppose (M, d) is a complete metric space and for each $n \in \mathbb{N}$, $T_n : M \to M$ is a (strict) contraction with contractivity coefficient ρ_n and fixed point x_n. Further suppose that (T_n) converges to $T : M \to M$ pointwise on M.

(a) Show that if (ρ_n) converges to ρ and $\rho < 1$, then T is also a (strict) contraction.

(b) Suppose that (x_n) converges to x^*. Prove that x^* is a fixed point of T. (You may **not** assume that (ρ_n) is convergent).

Q3 Consider the space $C[0, 1]$ of all continuous real valued functions on the interval $[0, 1]$, equipped with the uniform metric.

(a) State a necessary and sufficient condition for a set in $C[0, 1]$ to be compact.

(b) Suppose $\{a_n\}$ is a bounded sequence of real numbers. Show that the sequence $\{\sin(a_nx)\}$ has a subsequence that converges uniformly on $[0, 1]$.

(c) Show that the sequence $\{e^{nx}\}$ does not have a subsequence that converges uniformly on $[0, 1]$.

Q4 You are given the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2x^2 + nx}$$

on the domain $x \in (0, \infty)$.

(a) Prove that the series converges uniformly on $[a, \infty)$ for all $a > 0$.

(b) Explain why the sum is continuous on $(0, \infty)$.

(c) Discuss the differentiability of the sum on $(0, \infty)$.

(d) Does the series converge uniformly on $(0, \infty)$?

Q5 (a) Provide the definition of the Fréchet derivative of a map $F : V_1 \to V_2$ where $(V_i, \|.\|_i)$ are finite dimensional normed vector spaces.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x^4 + y^4}{x^2 + y^2}, \quad \text{if } y \text{ is irrational,}$$
$$f(x,y) = 0, \quad \text{if } y \text{ is rational.}$$

i. Decide if f has directional derivatives at $(0,0)$ along all $v \in \mathbb{R}^2$.

ii. Decide if f is Fréchet differentiable at $(0,0)$.

iii. What is the largest subset of $\mathbb{R}^2$ on which f is Fréchet differentiable?